摘 要

本篇論文所要探討的主題是：運用 Cox, Ingersoll and Ross (1985) 所建構的利率模型確保利率恆為正數，並計算當市場無套利下債券及其選擇權的價值。 我們以 explicit finite difference method 與 Monte Carlo method 針對美國 10 年債券及其歐式選擇權作實證分析。 前者為 Hull and White (1990) 提出，而後者為 Boyle (1977) 提出。 結果發現，以 explicit finite difference method 對美國 10 年債券及其歐式選擇權價格的評價計算之效率性較 Monte Carlo method 為佳。

Abstract

In this paper, we present how the CIR model guarantees interest rates against negative values in detail and what the prices of both discount bonds and European call options are when interest rates are assumed to follow the CIR model. In addition, we simulate European call option values on a U.S. 10-Y treasury bond in the CIR model by explicit finite difference and Monte Carlo methods. The former principle requires that the first two moments of both the modified and the real models be equal. It is presented in Hull and White (1990). The latter is used for the basic Monte Carlo method by Boyle (1977). Finally, we find that the effectiveness of the numerical computation by explicit finite difference is better than that by the basic Monte Carlo methods.

Keywords: CIR model, treasury bond, European option, explicit finite difference method, Monte Carlo method

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Notations

Symbol	Meaning
Ω	Sample space
$\mathcal{F}$	σ-algebra
P	Probability measure
$\mathbb{R}$	Real number
$W(\cdot)$	A standard Brownian motion
$E(\cdot)$	$E(X)$, the expectation of an r.v. X
$E(\cdot \mid \cdot)$	$E(X \mid Y)$, for all r.v. Y the conditional expectation of an r.v. X
$\wedge$	$a \wedge b$, minimum of a and b
$Var(\cdot)$	$Var(X)$, the variance of an r.v. X
$L^2[0,T]$	The measure defined by $\Big\{\{X(t),\mathcal{F}(t)\}_{t\in[0,T]}$ real value stochastic process : $\{X(t)\}_{t\in[0,T]}$ is $\mathcal{F}(\mathrm{t})-$measurable, and $\mathrm{E}\big(\int_0^{\mathrm{T}}\ \mathrm{X}^2(\mathrm{t})\ \mathrm{dt}\big) < \infty.\Big\}$
$I(\cdot)$	The *Itô Integral*
p	$p(t,T)$, the price at time t of a zero coupon bond with maturity date T
$B(\cdot)$	The money account process
$r(\cdot)$	The interest rate process
$\chi^2_{n,\delta}$	A non-central chi-square distribution with n d.f. and noncentral parameter δ
Γ	The Gamma function
V	The value of an European call option

Chapter 1

Introduction

In the last twenty years, interest-rate-contingent claims have become increasingly popular. The values of these securities are closely related to the shape and the stochastic movements of the term structure. Therefore, numerous models have been developed to simulate the interest rate movements.

The first appealing framework is due to Vasicek (1977) [21] who was the first to give an explicit characterization of the term structure. Vasicek proposed an Ornstein-Uhlenbeck process for the short interest rate. This process offers the interesting characteristic of mean reversion which is consistent with the observed market interest rate behavior. However, it has several drawbacks. The major disadvantage is that it can lead to negative rates.

The problem of negative rates was solved by Dothan (1978) [6], Courtadon (1982) [7] who proposed a one-factor lognormal model and also by Cox, Ingersoll and Ross (CIR) (1985) [5] who suggested a square root model. The latter leads to analytical solutions for the prices of both discount bonds and European call options. For this reason its use was widely spread in the market; and despite the publication of more consistent theoretical models, the CIR model is still respected as a benchmark for pricing interest rate claims.

In this paper, we present how the CIR model guarantees interest rates against negative values in detail and what the prices of both discount bonds and European call options are when interest rates are assumed to follow the CIR model. In addition, we simulate European call option values on a U.S. 10-Y treasury bond in the CIR model by explicit finite difference and Monte Carlo methods. The former principle requires that the first two moments of both the modified and the real models be equal. It is presented in Hull and

White (1990) [14]. The latter is used for the basic Monte Carlo method by Boyle (1977) [1].

The paper is organized as follows. In Chapter 2, we introduce basic ideas in stochastic processes and Brownian motion. Chapter 3 is an introduction to the bond market and the Cox-Ingersoll-Ross (CIR) interest rate model (1985) [5]. In Chapter 4, we show the results of using explicit finite difference and Monte Carlo methods to value a 10-Year bond. Finally, some concluding remarks are made in Chapter 5.

Chapter 2

Mathematical Preliminaries

Some mathematical relationships and stochastic concept in the interest rate model for later use are reviewed here.

2.1 Brownian Motion and Martingale

General assumptions

Let $(\Omega, \mathcal{F}, P)$ be a probability space with sample space Ω, σ-algebra $\mathcal{F}$ and probability measure P.

Definition 2.1 Let $X = \{X(t) : t \geq 0\}$ be a *stochastic process* if for all t, $X(t)$ is a random variable ; that is, $X(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$, $\{\omega : X(t, \omega) \in (a, b)\} \in \mathcal{F}$, for all $a < b$.

Definition 2.2 *A standard Brownian motion*, $W = \{W(t) : t \geq 0\}$ has the following properties :

1. $W(0) = 0$ a.s.; Technically, $P\{\omega : W(0, \omega) = 0\} = 1$,

2. $W(t)$ is a continuous function of t, for all ω,

3. If $0 = t_0 \leq t_1 \leq ... \leq t_n$, then the increments

$$W(t_1) - W(t_0), \cdots, W(t_n) - W(t_{n-1}) \overset{i.i.d.}{\sim} N(0, t_i - t_{i-1}),\ i = 1, 2, \cdots, n.$$

Definition 2.3 Let $\{\mathcal{F}(t) : t \in I\}$ be a family of sub-σ-algebra of $\mathcal{F}$, I be an ordered index set with $\mathcal{F}(s) \subset \mathcal{F}(t)$ for $s < t$, $s, t \in I$. Such a family $\{\mathcal{F}(t) : t \in I\}$ is called a *filtration*.

Definition 2.4 A stochastic process $X = \{X(t) : t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ is an $\mathcal{F}(t)$-measurable or $\mathcal{F}(t)$-adapted if

$$\sigma\Big(X(t)\Big) \subset \mathcal{F}(t),$$

that is,

$$\{X(t) \in (a, b)\} \subset \mathcal{F}(t), \ \forall \ a < b.$$

Definition 2.5 A stochastic process $X = \{X(t) : t = t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ is an P-martingale with respect to filtration $\{\mathcal{F}(t) : t \geq 0\}$ if

1. $E_P\Big(|X(t)|\Big) < \infty, \ t \geq 0,$
2. $X(t)$ is an $\mathcal{F}(t)$-adapted, $t \geq 0$,
3. $E_P\Big(X(t+1) \mid \mathcal{F}(t)\Big) = X(t), \ 0 \leq t \leq s.$

Theorem 2.1 *A standard Brownian motion $W = \{W(t) : t \geq 0\}$, is a martingale.*

Proof. If $s < t$, then $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ and

$$\begin{aligned} E\Big(W(t) \mid \mathcal{F}(s)\Big) &= E\Big(W(t) - W(s) + W(s) \mid \mathcal{F}(s)\Big) \\ &= E\Big(W(t) - W(s) \mid \mathcal{F}(s)\Big) + E\Big(W(s) \mid \mathcal{F}(s)\Big) \\ &= E\Big(W(t) - W(s)\Big) + W(s) \\ &= W(s). \end{aligned}$$

If $s \geq t$, then

$$E\Big(W(t) \mid \mathcal{F}(s)\Big) = W(t).$$

Thus

$$E\Big(W(t) \mid \mathcal{F}(s)\Big) = W(t \wedge s).$$

■

Definition 2.6 Let $\Pi = \{t_0, t_1, ..., t_n\}$ be a *partition* of $[0, T]$. The *mesh* of the partition is defined to be

$$\|\Pi\| = \max_{k=0,1,\cdots,n-1} (t_{k+1} - t_k).$$

We then define the *quadratic variation* of a function f on a interval $[0, T]$ is

$$\langle f \rangle(T) = \lim_{\|\Pi\| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2.$$

Theorem 2.2 *Let $W(t), t \geq 0$ be an standard Brownian motion. Then*

$$\langle W \rangle(T) = T,$$

or more precisely,

$$P\{\omega \in \Omega \ : \ \langle W(\cdot, \omega) \rangle(T) = T\} = 1.$$

Proof. Let $\Pi = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0, T]$. To simplify notation, set

$$D_k = W(t_{k+1}) - W \ (t_k)$$

and

$$\begin{aligned} Q_\Pi &= \sum_{k=0}^{n-1} D_k^2 \\ &= \sum_{k=0}^{n-1} |W(t_{k+1}) - W(t_k)|^2 \end{aligned}$$

We want to show

$$\lim_{\|\Pi\| \to 0} (Q_\Pi - T) = 0$$

Note

$$Q_\Pi - T = \sum_{k=0}^{n-1} \left[D_k^2 - (t_{k+1} - t_k)\right]$$

and

$$D_k = W(t_{k+1}) - W(t_k) \ \sim \ N(0, t_{k+1} - t_k), \ k = 0, 1, \cdots, n-1.$$

Then

$$\begin{aligned} E(Q_\Pi - T) &= E\Big(\sum_{k=0}^{n-1} \big[D_k^2 - (t_{k+1} - t_k)\big]\Big) \\ &= \sum_{k=0}^{n-1} \big[E(D_k^2) - (t_{k+1} - t_k)\big] \\ &= \sum_{k=0}^{n-1} \big[Var(D_k) - (t_{k+1} - t_k)\big] \\ &= 0. \end{aligned}$$

Since for $i \neq j$ D_i and D_j are independent, the terms

$$D_i^2 - (t_{k+1} - t_k) \text{ and } \mathrm{D_j^2 - (t_{k+1} - t_k)}$$

are also independent. Thus

$$\begin{aligned} Var(\ Q_\Pi - T\ ) &= Var\Big(\sum_{k=0}^{n-1} \big[D_k^2 - (t_{k+1} - t_k)\big]\Big) \\ &= \sum_{k=0}^{n-1} Var\Big(D_k^2 - (t_{k+1} - t_k)\Big) \\ &= \sum_{k=0}^{n-1} E\Big(\big[D_k^2 - (t_{k+1} - t_k)\big]^2\Big) \\ &= \sum_{k=0}^{n-1} E\Big(D_k^4 - 2(t_{k+1} - t_k){D_k}^2 + (t_{k+1} - t_k)^2\Big) \\ &= \sum_{k=0}^{n-1} \Big[E(D_k^4) - 2(t_{k+1} - t_k)E(D_k^2) + (t_{k+1} - t_k)^2\Big] \\ &= \sum_{k=0}^{n-1} \Big[3(t_{k+1} - t_k)^2 - (t_{k+1} - t_k)^2\Big] \end{aligned}$$

(if X is normal with mean 0 and variance σ^2, then $E(X^4) = 3\sigma^4$)

$$\begin{aligned} &= 2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \\ &\leq 2\ \|\Pi\| \sum_{k=0}^{n-1} (t_{k+1} - t_k) \\ &= 2\ \|\Pi\| T. \end{aligned}$$

We have

$$\begin{aligned} E(Q_\Pi - T) &= 0, \\ Var(Q_\Pi - T) &\leq 2\|\Pi\|T. \end{aligned}$$

As $\|\Pi\| \to 0, Var(Q_\Pi - T) \to 0$, so

$$\lim_{\|\Pi\|\to 0} (Q_\Pi - T) = 0.$$

■

Remark 2.1 We know that

$$E\Big[\big(W(t_{k+1}) - W(t_k)\big)^2 - (t_{k+1} - t_k)\Big] = 0.$$

We showed above that

$$Var\Big[\big(W(t_{k+1}) - W(t_k)\big)^2 - (t_{k+1} - t_k)\Big] = 2(t_{k+1} - t_k)^2.$$

When $(t_{k+1} - t_k)$ is small, $(t_{k+1} - t_k)^2$ is very small, and we have the approximate equation

$$\big(W(t_{k+1}) - W(t_k)\big)^2 \simeq (t_{k+1} - t_k),$$

which we can write informally as

$$dW(t)dW(t) = dt.$$

2.2 The Itô Calculus

The Itô Calculus is described for a class of processes known as *Itô Integral* which we now define.

Definition 2.7 Define the measure

$$\begin{aligned} L^2[0,T] &:= L^2\Big([0,T], \Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, P\Big) \\ &:= \Big\{\{X(t), \mathcal{F}(t)\}_{t\in[0,T]} \text{ real value stochastic process } \Big| \\ &\qquad \{X(t)\}_{t\in[0,T]} \text{ is } \mathcal{F}(\mathrm{t})\text{−measurable, and } \mathrm{E}\big(\textstyle\int_0^{\mathrm{T}} \mathrm{X}^2(\mathrm{t})\ \mathrm{dt}\big) < \infty.\Big\} \end{aligned}$$

To define a norm on $L^2[0,T]$, we set

$$\|X\|_T^2 := E\Big(\int_0^T X^2(t)\ dt\Big).$$

This is the well-known L^2-norm.

Definition 2.8 Fix $T > 0$. Let $\delta \in L^2[0,T]$ be a process and let $W = \{W(t) : t \geq 0\}$, be a Brownian motion with associated filtration $\mathcal{F}(t), t \geq 0$, and the following properties :

1. $s \leq t \Longrightarrow \mathcal{F}(s) \subset \mathcal{F}(t)$,
2. $W(t)$ is $\mathcal{F}(t)$-adapted, $\forall\ t$,
3. For $t \leq t_1 \leq \cdots \leq t_n$, the increment $W(t_1) - W(t), \cdots, W(t_n) - W(t_{n-1})$ are independent of $\mathcal{F}(t)$.

Then we define the *Itô Integral*

$$I(t) = \int_0^t \delta(u)\ dW(u),\ t \geq 0\ \ . \tag{2.1}$$

2.2.1 The Itô integral of a simple process

Let $\Pi = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0,T]$. Assume that δ is constant on each subinterval $[t_k, t_{k+1}]$ (see Fig. 2.1). We call such δ a simple function. Then the Itô integral $I(t)$ can be given by :

$$I(t) = \begin{cases} \delta(t_0)\big[W(t) - W(t_0)\big], \quad 0 \leq t \leq t_1, \\ \delta(t_0)\big[W(t_1) - W(t_0)\big] + \delta(t_1)\big[W(t) - W(t_1)\big], \quad t_1 \leq t \leq t_2, \\ \vdots \\ \delta(t_0)\big[W(t_1) - W(t_0)\big] + \cdots + \delta(t_{n-1})\big[W(t) - W(t_{n-1})\big], \quad t_{n-1} \leq t \leq t_n. \end{cases}$$

In general, if $t_k \leq t \leq t_{k+1}$,

$$I(t) = \sum_{j=0}^{k-1} \delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \delta(t_k)\big[W(t) - W(t_k)\big].$$

$\delta(t) = \delta(t_2)$

$\delta(t) = \delta(t_0)$

$\delta(t) = \delta(t_3)$

$t_0 = 0$ t_1 t_2 t_3 $t_4 = T$

$\delta(t) = \delta(t_1)$

Figure 2.1: An simple function δ.

Adaptedness For each $t, I(t)$ is $\mathcal{F}(t)$-measurable.

Linearity If

$$I(t) = \int_0^t \delta(u)\ dW(u),\ \ J(t) = \int_0^t \gamma(u)\ dW(u)$$

then

$$I(t) \pm J(t) = \int_0^t \Big(\delta(u) \pm \gamma(u)\Big)\ dW(u)$$

and

$$cI(t) = \int_0^t c\delta(u)\ dW(u),$$

where c is constant.

Martingale $I(t)$ is a martingale.

We prove the martingale property for the simple process case.

Theorem 2.3 (***Martingale Property***)

$$I(t) = \sum_{j=0}^{k-1} \delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \delta(t_k)\big[W(t) - W(t_k)\big],\ t_k \le t \le t_{k+1}$$

is a martingale.

Proof. Let $0 \le s < t$ be given. We treat the more difficult case that s and t are in different subintervals, i.e., there are partition points t_ℓ and t_k such that $s \in [\ t_\ell, t_{\ell+1}]$ and

$t \in [t_k, t_{k+1}]$(see Fig. 2.2). Write

$$\begin{aligned} I(t) &= \sum_{j=0}^{\ell-1} \delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \delta(t_\ell)\big[W(t_{\ell+1}) - W(t_\ell)\big] \\ &\quad + \sum_{j=\ell+1}^{k-1} \delta(t_j)\big[W(t_{j+1}) - W(t_j)\big] + \delta(t_k)\big[W(t) - W(t_k)\big]. \end{aligned}$$

We compute conditional expectations :

s t

t_ℓ $t_{\ell+1}$ $\cdots$ t_k t_{k+1}

Figure 2.2: Showing s and t in different partitions.

$$\begin{aligned} E\Big[\sum_{j=0}^{\ell-1} \delta(t_j)\big(W(t_{j+1}) - W(t_j)\big) \mid \mathcal{F}(s)\Big] &= \sum_{j=0}^{\ell-1} \delta(t_j)\big(W(t_{j+1}) - W(t_j)\big), \\ E\Big[\delta(t_\ell)\big(W(t_{\ell+1}) - W(t_\ell)\big) \mid \mathcal{F}(s)\Big] &= \delta(t_\ell)\Big(E\big[W(t_{\ell+1}) \mid \mathcal{F}(s)\big] - W(t_\ell)\Big) \\ &= \delta(t_\ell)\big[W(t_s) - W(t_\ell)\big], \end{aligned}$$

and

$$\begin{aligned} & E\Big[\sum_{j=\ell+1}^{k-1} \delta(t_j)\big(W(t_{j+1}) - W(t_j)\big) \mid \mathcal{F}(s)\Big] \\ &= \sum_{j=\ell+1}^{k-1} E\Big[E\Big[\delta(t_j)\big(W(t_{j+1}) - W(t_j)\big) \mid \mathcal{F}(t_j)\Big] \mid \mathcal{F}(s)\Big] \\ &= \sum_{j=\ell+1}^{k-1} E\Big[\delta(t_j)\Big(E\big[W(t_{j+1}) \mid \mathcal{F}(t_j)\big] - W(t_j)\Big) \mid \mathcal{F}(s)\Big] \\ &= \sum_{j=\ell+1}^{k-1} E\Big[\delta(t_j)\big(W(t_j) - W(t_j)\big) \mid \mathcal{F}(s)\Big] \\ &= 0, \end{aligned}$$

$$E\Big[\delta(t_k)\big(W(t)-W(t_k)\big) \;\Big|\; \mathcal{F}(s)\Big]$$

$$= E\Big[\delta(t_k)\Big(E\big[W(t) \mid \mathcal{F}(t_k)\big]-W(t_k)\Big) \;\Big|\; \mathcal{F}(s)\Big]$$

$$= E\Big[\delta(t_k)\big(W(t_k)-W(t_k)\big) \;\Big|\; \mathcal{F}(s)\Big]$$

$$= 0.$$

Then

$$\begin{aligned} E\big[I(t) \mid \mathcal{F}(s)\big] &= \sum_{j=0}^{\ell-1}\Big(\delta(t_j)\big(W(t_{j+1})-W(t_j)\big)\Big)+\delta(t_\ell)\big(W(t_s)-W(t_\ell)\big) \\ &= I(s). \end{aligned}$$

■

Theorem 2.4 (***Itô Isometry***)

$$E\big[I^2(t)\big] = E\big[\int_0^t \delta^2(u)\, du\big].$$

Proof. To simplify notation, assume $t = t_k$, so

$$I(t) = \sum_{j=0}^{k} \delta(t_j)\big[W(t_{j+1})-W(t_j)\big]$$

Each $W(t_{j+1}) - W(t_j)$ has expectation 0, and different $W(t_{j+1}) - W(t_j)$ are independent.

$$\begin{aligned} I^2(t) &= \Big(\sum_{j=0}^{k} \delta(t_j)\big[W(t_{j+1})-W(t_j)\big]\Big)^2 \\ &= \sum_{j=0}^{k} \delta^2(t_j)\big[W(t_{j+1})-W(t_j)\big]^2 \\ &\quad +2\sum_{i<j} \delta(t_i)\ \delta(t_j)\ \big[W(t_{i+1})-W(t_i)\big]\ \big[W(t_{j+1})-W(t_j)\big]. \end{aligned}$$

Since the cross terms have expectation zero,

$$\begin{aligned} E\big[I^2(t)\big] &= \sum_{j=0}^{k} E\Big[\delta^2(t_j)\big(W(t_{j+1}) - W(t_j)\big)^2\Big] \\ &= \sum_{j=0}^{k} E\Big[\delta^2(t_j)\ (t_{j+1} - t_j)\Big] \quad \text{(by Remark 2.1)} \\ &= E\Big[\sum_{j=0}^{k} \Big(\int_{t_j}^{t_{j+1}} \delta^2(u)\ du\Big)\Big] \\ &= E\Big[\int_0^t \delta^2(u)\ du\Big] \end{aligned}$$

■

2.2.2 The general Itô integral

Let δ be a process (not necessarily a simple process). We now define

$$\int_0^T \delta(t)\ dW(t) = \lim_{n\to\infty} \int_0^T \delta_n(t)\ dW(t),$$

where $\{\delta_n\}_{n=1}^{\infty}$ is a sequence of simple processes.

The only difficulty with this approach is that we need to make sure the above limit exists. To proof the above limit exists, we are in need of Theorem below.

Theorem 2.5 *An arbitrary $\delta \in L^2[0,T]$ can be approximated by a sequence of simple processes δ_n. More precisely: There exists a sequence of simple processes $\{\delta_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} E\Big(\int_0^T \big(\delta_n(t) - \delta(t)\big)^2\ dt\Big) = 0.$$

Proof. Define

$$\delta_n = \sum_{k=1}^{n2^n} \Big(\frac{k-1}{2^n}\Big) 1_{[\frac{k-1}{2^n} \le \delta < \frac{k}{2^n}]} + n1_{[\delta \ge n]},$$

where 1 is an indicator function. Because $\delta \in L^2[0,T]$, it follows that $\delta_n \in L^2[0,T]$ and δ_n is a sequence of simple processes(see Fig. 2.3 and Fig. 2.4).

Note

$$\delta_n \le \delta_{n+1}.$$

If $\delta(\omega) < \infty$, then for all large enough n

$$\mid \delta(\omega) - \delta_n(\omega) \mid \leq \frac{1}{2^n} \to 0.$$

If $\delta(\omega) = \infty$, then $\delta_n(\omega) = n \to \infty$. Since L^2-norm is defined by

$$\|\delta\|_T^2 := E\Big(\int_0^T \delta^2(t)\ dt\Big) < \infty,$$

then

$$\lim_{n\to\infty} E\Big(\int_0^T \big(\delta_n(t) - \delta(t)\big)^2\ dt\Big) = 0.$$

■

We have defined

$$I_n(T) = \int_0^T \delta_n(t)\ dW(t),$$

for every n. Suppose n and m are large positive integers. Then

$$\begin{aligned}
Var\big[I_n(T) - I_m(T)\big] &= E\Big[\Big(\textstyle\int_0^T \big[\delta_n(t) - \delta_m(t)\big]\ dW(t)\Big)^2\Big] \\
&= E\Big[\textstyle\int_0^T \big[\delta_n(t) - \delta_m(t)\big]^2\ dt\Big] \quad \text{(by Itô Isometry)} \\
&= E\Big[\textstyle\int_0^T \big(\mid \delta_n(t) - \delta(t) + \delta(t) - \delta_m(t) \mid \big)^2\ dt\Big] \\
&\leq 2\ E\Big[\textstyle\int_0^T \mid \delta_n(t) - \delta(t) \mid^2\ dt\Big] + 2\ E\Big[\textstyle\int_0^T \mid \delta_m(t) - \delta(t) \mid^2\ dt\Big], \\
&\quad \text{(by } (\mathrm{a}+\mathrm{b})^2 \leq 2\mathrm{a}^2 + 2\mathrm{b}^2)
\end{aligned}$$

δ_1
1

1/2

0 1/4 1/2 3/4 1 5/4 3/2 7/4 2 δ

Figure 2.3: $n = 1$

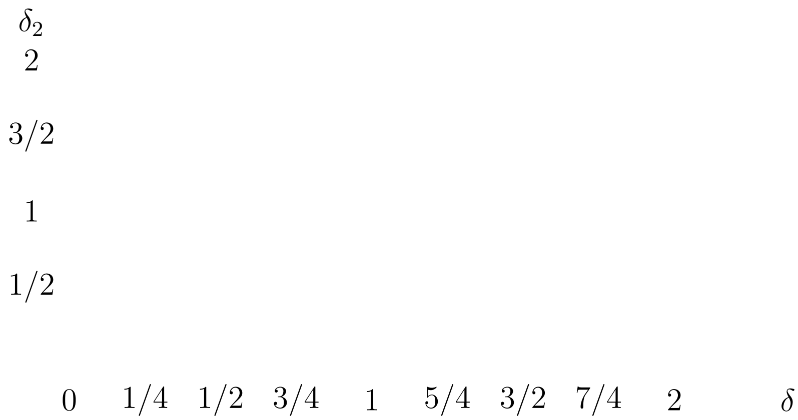

Figure 2.4: $n = 2$

which is small. This guarantees that the sequence $\{I_n(T)\}_{n=1}^{\infty}$ has a limit.

We now define

$$I(t) = \int_0^t \delta(u)\ dW(u),$$

where δ is any adapted, square-integrable process.

Adaptedness For each $t, I(t)$ is $\mathcal{F}(t)$-measurable.

Linearity If

$$I(t) = \int_0^t \delta(u)\ dW(u),\ \ J(t) = \int_0^t \gamma(u)\ dW(u)$$

then

$$I(t) \pm J(t) = \int_0^t \big(\delta(u) \pm \gamma(u)\big)\ dW(u)$$

and

$$cI(t) = \int_0^t c\delta(u)\ dW(u),$$

where c is constant.

Martingale $I(t)$ is a martingale.

Itô Isometry $E\big[I^2(t)\big] = E\big[\int_0^t \delta^2(u)\ du\big]$.

2.2.3 Itô Formula

We want a rule to "differentiate" expressions of the form $f(W(t))$, where $f(x)$ is a differentiable function. If $W(t)$ were also differentiable, then the ordinary *chain rule* would give

$$\frac{d}{dt} f\big(W(t)\big) = f'\big(W(t)\big)\ W'(t),$$

which could be written in differential notation as

$$\begin{aligned} df\big(W(t)\big) &= f'\big(W(t)\big)\ W'(t)\ dt \\ &= f'\big(W(t)\big)\ dW(t). \end{aligned}$$

However, $W(t)$ is not differentiable, and in particular has nonzero quadratic variation, so the correct formula has an extra term, namely,

$$\begin{aligned} df\big(W(t)\big) &= f'\big(W(t)\big)\ dW(t) + \tfrac{1}{2} f''\big(W(t)\big)\ \big(dW(t)\big)^2 \\ &= f'\big(W(t)\big)\ dW(t) + \tfrac{1}{2} f''\big(W(t)\big)\ dt \quad \text{(by Remark 2.1)} \end{aligned}$$

This is *Itô formula in differential form.* Integrating this, we obtain *Itô formula in integral form*:

$$f\big(W(t)\big) - f\big(W(0)\big) = \int_0^t f'\big(W(u)\big)\ dW(u) + \frac{1}{2} \int_0^t f''\big(W(u)\big)\ du.$$

Definition 2.9 (**Geometric Brownian Motion**) Geometric Brownian motion is

$$S(t) = S(0) \exp\Big\{\sigma W(t) + \big(\mu - \frac{1}{2}\sigma^2\big) t\Big\},$$

where μ and $\sigma > 0$ are constant.

Define

$$f(t, x) = S(0) \exp\Big\{\sigma x + \big(\mu - \frac{1}{2}\sigma^2\big) t\Big\},$$

so

$$S(t) = f\big(t, W(t)\big).$$

Since

$$f_t = \big(\mu - \frac{1}{2}\sigma^2\big) f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f,$$

according to Itô's formula,

$$\begin{aligned} dS(t) &= df\big(t, W(t)\big) \\ &= \big(\mu - \tfrac{1}{2}\sigma^2\big) f \ dt + \sigma f \ dW(t) + \tfrac{1}{2}\sigma^2 f \ dt \\ &= \mu S(t) \ dt + \sigma S(t) \ dW(t). \end{aligned}$$

Thus, *Geometric Brownian motion in differential form* is

$$dS(t) = \mu S(t) \ dt + \sigma S(t) \ dW(t),$$

and *Geometric Brownian motion in integral form* is

$$S(t) = S(0) + \int_0^t \mu S(u) \ du + \int_0^t \sigma S(u) \ dW(u).$$

By Itô's formula, we also get the import theorem below.

Theorem 2.6 *Let $W(t)$ be a Brownian motion and $\delta(t)$ a nonrandom function. Then the Itô Integral*

$$I(t) = \int_0^t \delta(u) \ dW(u), \ t \geq 0$$

is a Gaussian process with its mean function $m(t) = 0$ and its covariance function

$$\rho(s,t) = \int_0^{s \wedge t} \delta^2(u) \ du, \ s \geq 0, \ t \geq 0.$$

Proof. Note

$$dI(t) = \delta(t) dW(t).$$

By Itô's formula, for $\theta \in \mathbb{R}$,

$$\begin{aligned}
de^{\theta I(s)} &= e^{\theta I(s)}\theta \; dI(s) + \frac{1}{2!}e^{\theta I(s)}\theta^2 \; \big(dI(s)\big)^2, \\
&= \theta e^{\theta I(s)}\delta(s) \; dW(s) + \frac{1}{2}\theta^2 e^{\theta I(s)}\delta^2(s) \; ds, \\
e^{\theta I(s)} &= e^{\theta I(0)} + \theta \int_0^s e^{\theta I(u)}\delta(u) \; dW(u) + \frac{1}{2}\theta^2 \int_0^s e^{\theta I(u)}\delta^2(u) \; du, \\
E\big(e^{\theta I(s)}\big) &= e^{\theta I(0)} + \frac{1}{2}\theta^2 \int_0^s E\big(e^{\theta I(u)}\big)\delta^2(u) \; du, \\
\frac{d}{ds}E\big(e^{\theta I(s)}\big) &= \frac{1}{2}\theta^2\delta^2(s)E\big(e^{\theta I(s)}\big), \\
E\big(e^{\theta I(s)}\big) &= e^{\theta I(0)} \exp\Big\{\frac{1}{2}\theta^2 \int_0^s \delta^2(u) \; du\Big\} \\
&= \exp\Big\{\frac{1}{2}\theta^2 \int_0^s \delta^2(u) \; du\Big\}.
\end{aligned}$$

This show that $I(s)$ is normal with mean 0 and variance $\int_0^s \delta^2(u) \; du$.

Let $0 \le s \le t$.

By Itô's formula,

$$de^{\theta I(t)} = \theta e^{\theta I(t)}\delta(t) \; dW(t) + \frac{1}{2}\theta^2 e^{\theta I(t)}\delta^2(t) \; dt.$$

Integrate form s to t to get

$$e^{\theta I(t)} = e^{\theta I(s)} + \theta \int_s^t e^{\theta I(u)}\delta(u) \; dW(u) + \frac{1}{2}\theta^2 \int_s^t e^{\theta I(u)}\delta^2(u) \; du.$$

Note

$$\begin{aligned}
&E\Big(\int_s^t e^{\theta I(u)}\delta(u) \; dW(u) \; \Big| \; \mathcal{F}(s)\Big) \\
&= E\Big(\int_0^t e^{\theta I(u)}\delta(u) \; dW(u) \; \Big| \; \mathcal{F}(s)\Big) - \int_0^s e^{\theta I(u)}\delta(u) \; dW(u) \\
&= 0.
\end{aligned}$$

Then

$$
\begin{aligned}
E\Big(e^{\theta I(t)} \Bigm| \mathcal{F}(s)\Big) &= e^{\theta I(s)} + \frac{1}{2}\theta^2 \int_s^t E\Big(e^{\theta I(u)} \Bigm| \mathcal{F}(s)\Big)\delta^2(u)\ du,\\
\frac{d}{dt}E\Big(e^{\theta I(t)} \Bigm| \mathcal{F}(s)\Big) &= \frac{1}{2}\theta^2\delta^2(t)E\Big(e^{\theta I(t)} \Bigm| \mathcal{F}(s)\Big),\ t \geq s,\\
E\Big(e^{\theta I(t)} \Bigm| \mathcal{F}(s)\Big) &= e^{\theta I(s)} \exp\Big\{\frac{1}{2}\theta^2 \int_s^t \delta^2(u)\ du\Big\},\ t \geq s.
\end{aligned}
$$

Thus the moment generating function for $\big(I(s), I(t)\big), 0 \leq s \leq t$, is

$$
\begin{aligned}
&E\Big(e^{\theta_1 I(s)+\theta_2 I(t)} \Bigm| \mathcal{F}(s)\Big)\\
&= e^{\theta_1 I(s)} E\Big(e^{\theta_2 I(t)} \Bigm| \mathcal{F}(s)\Big)\\
&= e^{(\theta_1+\theta_2)I(s)} \exp\Big\{\frac{1}{2}\theta_2^2 \int_s^t \delta^2(u)\ du\Big\},\\
&E\Big(e^{\theta_1 I(s)+\theta_2 I(t)}\Big)\\
&= E\Big(E\big(e^{\theta_1 I(s)+\theta_2 I(t)} \bigm| \mathcal{F}(s)\big)\Big)\\
&= E\Big(e^{(\theta_1+\theta_2)I(s)} \exp\Big\{\frac{1}{2}\theta_2^2 \int_s^t \delta^2(u)\ du\Big\}\Big)\\
&= \exp\Big\{\frac{1}{2}(\theta_1+\theta_2)^2 \int_0^s \delta^2(u)\ du\Big\} \exp\Big\{\frac{1}{2}\theta_2^2 \int_s^t \delta^2(u)\ du\Big\}\\
&= \exp\Big\{\frac{1}{2}(\theta_1^2+2\theta_1\theta_2) \int_0^s \delta^2(u)\ du + \frac{1}{2}\theta_2^2 \int_0^t \delta^2(u)\ du\Big\}\\
&= \exp\left\{\frac{1}{2}\,[\theta_1 \quad \theta_2] \begin{bmatrix} \int_0^s \delta^2(u)\ du & \int_0^s \delta^2(u)\ du \\ \int_0^s \delta^2(u)\ du & \int_0^t \delta^2(u)\ du \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}\right\}.
\end{aligned}
$$

This show that $\big(I(s), I(t)\big)$ is jointly normal with

$$E\big(I(s)\big) = E\big(I(t)\big) = 0,$$

$$E\big(I^2(s)\big) = \int_0^s \delta^2(u)\ du,$$

$$E\big(I^2(t)\big) = \int_0^t \delta^2(u)\ du,$$

$$E\big(I(s)I(t)\big) = \int_0^s \delta^2(u)\ du.$$

■

Remark 2.2 By Theorem 2.5 and Theorem 2.6 , we known when $\delta(t)$ a deterministic function, the Itô Integral $I(t)$ is also a Gaussian process with its mean function $m(t) = 0$ and its covariance function

$$\rho(s,t) = \int_0^{s\wedge t} \delta^2(u)\ du.$$

2.3 Change of measure - Cameron-Martin-Girsanov Theorem

Theorem 2.7 (***Cameron-Martin-Girsanov Theorem***) *Suppose $W(t), 0 \leq t \leq T$, is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}(t), 0 \leq t \leq T$, is a filtration, and $\lambda(t) \in L^2[0,T]$. Define a new measure Q by*

$$Q(A) = \int_A M\ dP,\ \forall\ A \in \mathcal{F},$$

where $M(t) = \exp\Big(-\int_0^t \lambda(s)\ dW(s) - \frac{1}{2}\int_0^t \lambda^2(s)\ ds\Big)$.
Then the process

$$\widetilde{W}(t) = W(t) + \int_0^t \lambda(s)\ ds$$

is a Q Brownian motion.

To prove Cameron-Martin-Girsanov Theorem, we are in need of some Theorem, Lemma and Remark below.

Theorem 2.8 *Let $M = \big\{M(t) : t \in [0,T]\big\}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ and*

$$M(t) = \exp\Big(-\int_0^t \lambda(s)\ dW(s) - \frac{1}{2}\int_0^t \lambda^2(s)\ ds\Big).$$

Then $M(t)$ is a $\mathcal{F}(t)$-martingale under P.

Proof. In fact,

$$\begin{aligned} dM(t) &= M(t)\cdot\big[-\lambda(t)\ dW(t) - \tfrac{1}{2}\lambda^2(t)\ dt + \tfrac{1}{2}\lambda^2(t)\ dt\big] \\ &= -\lambda(t)M(t)\ dW(t). \end{aligned}$$

Then

$$\begin{aligned} E_P\big(dM(t)\ \big|\ \mathcal{F}_t\big) &= E_P\big(-\lambda(t)M(t)\ dW(t)\ \big|\ \mathcal{F}(t)\big) \\ &= -\lambda(t)M(t)E_P\big(dW(t)\ \big|\ \mathcal{F}(t)\big) \\ &= 0. \end{aligned}$$

■

Remark 2.3 In Fig.2.5, Fig.2.6, and Fig.2.7, we take $\lambda(s) = 1$, $\lambda(s) = s^2 + 2s + 3$, and $\lambda(s) = cos(s/30)$, respectively. If t are large, then $E_P\big(dM(t)\ \big|\ \mathcal{F}_t\big) = 0$ a.s. . This shows that $M(t)$ is a $\mathcal{F}$-martingale under P.

Remark 2.4 The new measure Q described in Cameron-Martin-Girsanov Theorem is a probability space. For all $A \in \mathcal{F}$,

$$\begin{aligned} Q(A) &= \int_A M_T(\omega)\ dP(\omega) \\ &= \int_\Omega 1_A M_T(\omega)\ dP(\omega) \\ &= E_P(1_A M(T)), \end{aligned}$$

where 1 is an indicator function. Since $M(t) \geq 0$ for all $t \geq 0$, we have $Q(A) \geq 0$ for all $A \in \mathcal{F}$. If $A, B \in \mathcal{F}, A \cap B = \emptyset$, then

$$\begin{aligned} Q(A \cup B) &= \int_{A\cup B} M_T(\omega)\ dP(\omega) \\ &= \int_A M_T(\omega)\ dP(\omega) + \int_B M_T(\omega)\ dP(\omega) \\ &= Q(A) + Q(B). \end{aligned}$$

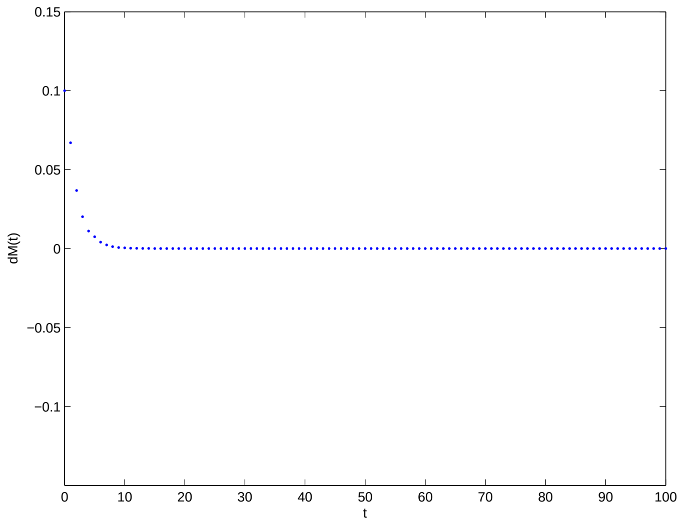

Figure 2.5: $\lambda = 1$

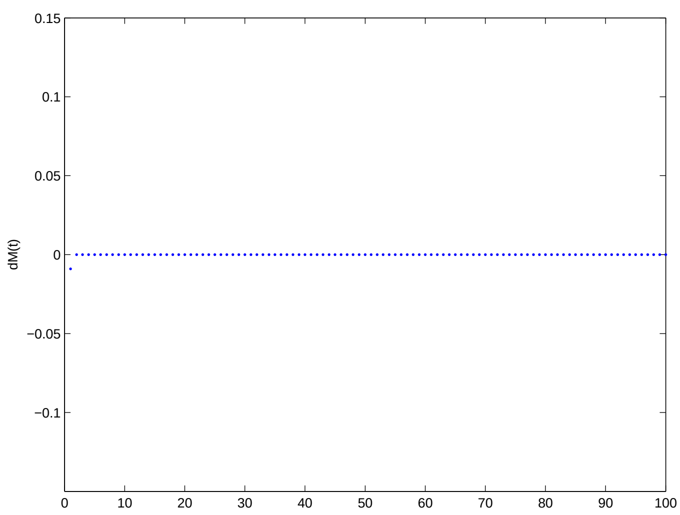

Figure 2.6: $\lambda(s) = s^2 + 2s + 3$

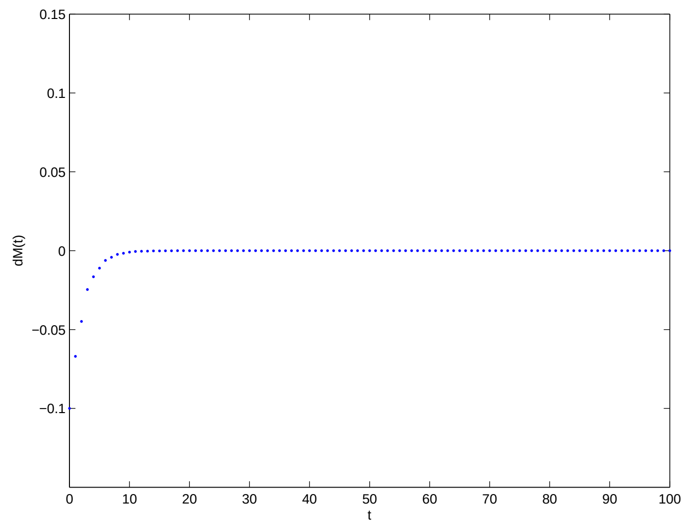

Figure 2.7: $\lambda(s) = cos(s/30)$

And we have

$$\begin{aligned} Q(\Omega) &= \int_\Omega M_T(\omega)\ dP(\omega) \\ &= E_P\big(M(T)\big) \\ &= E_P\big(M(T)\ \big|\ \mathcal{F}(0)\big) \\ &= M(0) \\ &= 1. \end{aligned}$$

Remark 2.5 For all $A \in \mathcal{F}$,

$$Q(A) = \int_A dQ(\omega) = \int_\Omega 1_A(\omega)\ dQ(\omega) = E_Q(1_A).$$

Also,

$$Q(A) = \int_\Omega 1_A M_T(\omega)\ dP(\omega) = E_P\big(1_A M(T)\big).$$

Thus

$$E_Q\big(1_A\big) = E_P\big(1_A M(T)\big).$$

Lemma 2.1 *Let $0 \le t \le T$. If X is $\mathcal{F}(t)$-measurable, then*

$$E_Q(X) = E_P(XM(t)).$$

Proof. Since $M(t), 0 \le t \le T$, is a martingale under P, then

$$\begin{aligned} E_Q(X) &= E_P\big(XM(T)\big) = E_P\Big[E_P\Big(XM(T) \mid \mathcal{F}(t)\Big)\Big| \mathcal{F}(0)\Big] \\ &= E_P\Big[X \cdot E_P\big(M(T) \mid \mathcal{F}(t)\big)\Big] = E_P\big(XM(t)\big). \end{aligned}$$

■

Lemma 2.2 (***Baye's Rule***) *If X is $\mathcal{F}(t)$-measurable and $0 \le s \le t \le T$, then*

$$E_Q\big(X \mid \mathcal{F}(s)\big) = \frac{1}{M(s)} \cdot E_P\big(XM(t) \mid \mathcal{F}(s)\big).$$

Proof. For $A \in \mathcal{F}(s) \subset \mathcal{F}(t)$, we have

$$\begin{aligned} & E_Q\Big[1_A \frac{1}{M(s)} E_P\big(XM(t) \mid \mathcal{F}(s)\big)\Big] \\ = \; & E_P\Big[1_A E_P\big(XM(t) \mid \mathcal{F}(s)\big)\Big] \qquad (\text{ by Lemma 2.1}) \\ = \; & E_P\Big[1_A X M(s)\Big] \qquad (\text{ by Theorem 2.8}) \\ = \; & E_Q\Big[1_A X] \qquad (\text{ by Lemma 2.1 again}) \\ = \; & E_Q\Big[E_Q\big(1_A X \mid \mathcal{F}(s)\big)] \\ = \; & E_Q\Big[1_A E_Q\big(X \mid \mathcal{F}(s)\big)\Big] \end{aligned}$$

Thus

$$E_Q\big(X \mid \mathcal{F}(s)\big) = \frac{1}{M(s)} \cdot E_P\big(XM(t) \mid \mathcal{F}(s)\big).$$

■

Theorem 2.9 *Using the description of Cameron-Martin-Girsanov Theorem, we have the martingale property*

$$E_Q\big(\widetilde{W}(t) \mid \mathcal{F}(s)\big) = \widetilde{W}(s), \; 0 \le s \le t \le T.$$

Proof. We first check that $\widetilde{W}(t)M(t)$ is a martingale under P. Recall

$$\begin{aligned} d\widetilde{W}(t) &= \lambda(t)\ dt + dW(t), \\ dM(t) &= -\lambda(t)M(t)\ dW(t). \end{aligned}$$

Then

$$\begin{aligned} d(\widetilde{W}(t)M(t)) &= \widetilde{W}(t)\ dM(t) + M(t)\ d\widetilde{W}(t) + d\widetilde{W}(t)\ dM(t) \\ &= -\widetilde{W}(t)\lambda(t)M(t)\ dW(t) + M(t)\lambda(t)\ dt + M(t)\ dW(t) - \lambda(t)M(t)\ dt \\ &= (-\widetilde{W}(t)\lambda(t)M(t) + M(t))\ dW(t). \end{aligned}$$

Therefore, for $0 \le s \le t \le T$,

$$\begin{aligned} E_P\Big(d\big(\widetilde{W}(t)M(t)\big) \mid \mathcal{F}(s)\Big) &= E_P\Big(\big(-\widetilde{W}(t)\lambda(t)M(t) + M(t)\big)E_P\big(dW(t) \mid \mathcal{F}(t)\big)\Big|\ \mathcal{F}(s)\Big) \\ &= 0. \end{aligned}$$

Next we use Baye's Rule. For $0 \le s \le t \le T$,

$$\begin{aligned} E_Q\big[\widetilde{W}(t) \mid \mathcal{F}(s)\big] &= \frac{1}{M(s)}E_P\big(\widetilde{W}(t)M(t) \mid \mathcal{F}(s)\big) \\ &= \frac{1}{M(s)}\widetilde{W}(s)M(s) \\ &= \widetilde{W}(s). \end{aligned}$$

■

proof of Cameron-Martin-Girsanov Theorem .

To show the process $\widetilde{W}(t), 0 \le t \le T$, is a standard Brownian motion we verify it satisfies Definition 2.2 . First,

$$\widetilde{W}(0) = W(0) + \int_0^0 \lambda(u)\ du = 0.$$

Second, since $W(t)$ is continuous a.s. and an indefinite integral is a continuous process, then $\widetilde{W}(t)$ is a continuous process a.s. . Finally, take

$$X(t) = \int_0^t (\theta - \lambda(u))\ dW(u),\ \theta \in \mathbb{R}.$$

Then

$$\begin{aligned}
de^{X(t)} &= e^{X(t)}\Big[\big(\theta-\lambda(t)\big)\ dW(t)+\tfrac{1}{2}\big(\theta-\lambda(t)\big)^2\ dt\Big] \\
e^{X(t)} &= e^{X(0)}+\int_0^t \big(\theta-\lambda(u)\big)e^{X(u)}\ dW(u)+\frac{1}{2}\int_0^t \big(\theta-\lambda(u)\big)^2 e^{X(u)}\ du \\
E_P(\ e^{X(t)}\ ) &= 1+\frac{1}{2}\int_0^t \big(\theta-\lambda(u)\big)^2\ E_P(e^{X(u)})\ du \\
\frac{d}{dt}E_P(e^{X(t)}) &= \tfrac{1}{2}\big(\theta-\lambda(t)\big)^2\ E_P(e^{X(t)}) \\
E_P(\ e^{X(t)}\ ) &= e^{X(0)}\exp\Big\{\frac{1}{2}\int_0^t \big(\theta-\lambda(u)\big)^2\ du\Big\} \\
&= \exp\Big\{\frac{1}{2}\int_0^t \big(\theta-\lambda(u)\big)^2\ du\Big\}.
\end{aligned}$$

For all $t\in[0,T]$, the moment generating function of $W(t)$ under Q is

$$\begin{aligned}
E_Q\big[e^{\theta W(t)}\big] &= E_P\big[M(t)e^{\theta W(t)}\big] \\
&= E_P\Big[\exp\Big\{-\int_0^t \lambda(u)\ dW(u)-\frac{1}{2}\int_0^t \lambda^2(u)\ du\Big\}\cdot\exp\Big\{\int_0^t \theta\ dW(u)\Big\}\Big] \\
&= \exp\Big\{-\frac{1}{2}\int_0^t \lambda^2(u)\ du\Big\}\ E_P\Big[\exp\Big\{\int_0^t \big(\theta-\lambda(u)\big)\ dW(u)\Big\}\Big] \\
&= \exp\Big\{-\frac{1}{2}\int_0^t \lambda^2(u)\ du\Big\}\cdot\exp\Big\{\frac{1}{2}\int_0^t \big(\theta-\lambda(u)\big)^2\ du\Big\} \\
&= \exp\Big\{-\theta\int_0^t \lambda(u)\ du+\frac{1}{2}\theta^2\int_0^t 1\ du\Big\}
\end{aligned}$$

Thus

$$W(t)\ \sim\ N_Q(-\int_0^t \lambda(u)\ du, t).$$

That is ,

$$\widetilde{W}(t)=W(t)+\int_0^t \lambda(u)\ du\ \sim\ N_Q(0,t).$$

By Theorem 2.9, we have $\widetilde{W}(t)$ is a Q-martingale. Thus for $0=t_0\le t_1\le\cdots\le t_n=T$, then the increments

$$\widetilde{W}(t_1)-\widetilde{W}(t_0),\cdots,\widetilde{W}(t_n)-\widetilde{W}(t_{n-1})$$

are independent, normal, and

$$E_Q\big[\widetilde{W}(t_i) - \widetilde{W}(t_{i-1})\big] = 0,$$

$$E_Q\big[\big(\widetilde{W}(t_i) - \widetilde{W}(t_{i-1})\big)^2\big] = t_i - t_{i-1},\ i = 1, 2, \cdots, n.$$

■

Remark 2.6 In Fig.2.8, Fig.2.9, and Fig.2.10, we take $\lambda(s) = 1$, $\lambda(s) = s^2 + 2s + 3$, and $\lambda(s) = cos(s/30)$, respectively. The process $\widetilde{W}$ isn't a P Brownian motion, it has a shift that dependents on λ.

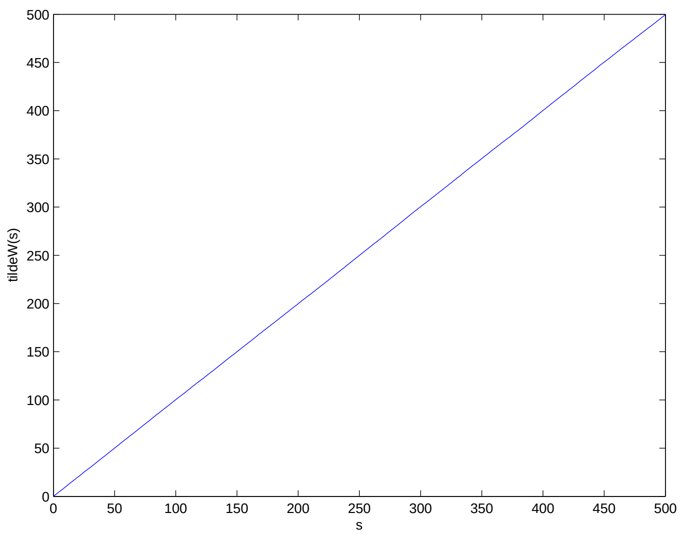

Figure 2.8: $\lambda = 1$

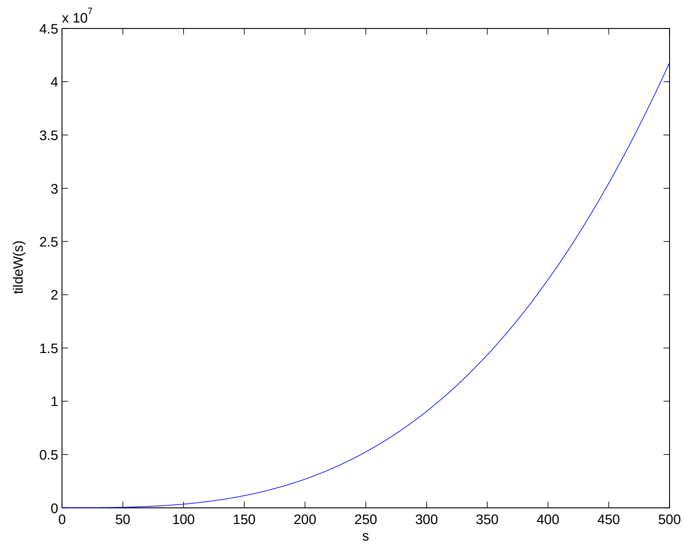

Figure 2.9: $\lambda(s) = s^2 + 2s + 3$

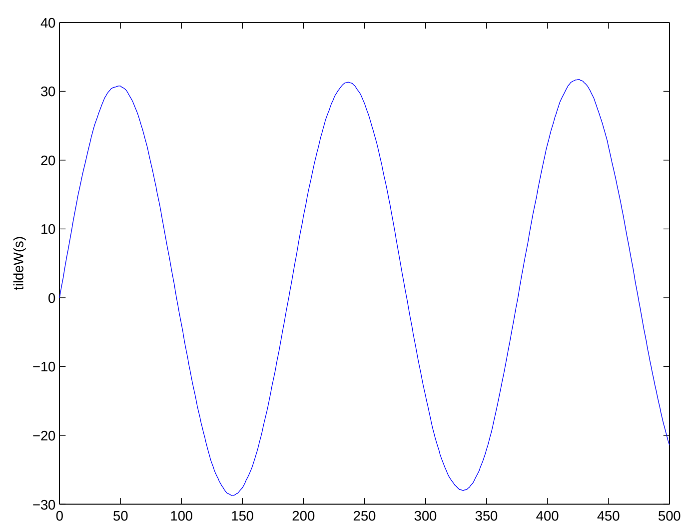

Figure 2.10: $\lambda(s) = cos(s/30)$

Chapter 3

Bonds and the Cox-Ingersoll-Ross Model

3.1 Generalities

Since we can treat a coupon bond as a linear combination of zero coupon bonds for each time of maturity, then the bond market we will study is mainly the market of zero coupon bond. Given a filtered probability space $(\Omega, \mathcal{F}, P)$.

Definition 3.1 A *zero coupon bond* with maturity date T, also called a T-bond, is a contract which guarantees the holder one dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t, T)$.

We now make an assumption to guarantee the existence of a sufficiently rich bond market.

Assumption 3.1

1. For every $T > 0$, there exists a market for T-bonds.

2. For every fixed T, the process $\{p(t) : 0 \leq t \leq T\}$ is an optional stochastic process with $p(t, t) = 1$ for all t.

3. For every fixed t, $p(t, T)$ is P-a.s. continuously differentiable in the T-variable. This partial derivative is often denoted by

$$p_T(t, T) = \frac{\partial p(t, T)}{\partial T}.$$

Given the bond market above, we may now define a number of interest rates, and the basic construction is as follows. Suppose that we are standing at time t, and let us fix two other points in time, T and $T+\varepsilon$, with $t < T < T+\varepsilon$. The immediate project is to write a contract at time t which allows us to make an investment of one dollar at time T, and to have a deterministic rate of return, determined at the contract time t, over the interval $[T, T+\varepsilon]$. This can easily be achieved as follows.

1. At time t we sell one T-bond. This will give us $p(t,T)$ dollars.
2. We use this income to buy exactly $p(t,T)/p(t,T+\varepsilon)$ T-bonds. Thus our net investment at time t equals zeros;

$$p(t,T) - \frac{p(t,T)}{p(t,T+\varepsilon)} p(t,T+\varepsilon) = 0.$$

3. At time T the T-bond matures, so we are obliged to pay out one dollar.
4. At time $T+\varepsilon$ the $T+\varepsilon$-bonds mature at one dollar a piece, so we will receive the amount $p(t,T)/p(t,T+\varepsilon)$ dollars.
5. The net effect of all this is that, based on a contract at t, an investment of one dollar at time T has yielded $p(t,T)/p(t,T+\varepsilon)$ at time $T+\varepsilon$. We now determine the equivalent constant short rate of interest over this period as the solution R to the equation

$$\frac{p(t,T)}{p(t,T+\varepsilon)} = \exp\{\varepsilon \cdot R(t,T,T+\varepsilon)\} \cdot 1.$$

Based on this argument we proceed to the formal definitions.

Definition 3.2

1. The **forward rate** for $[T, T+\varepsilon]$ contracted at t is defined as

$$R(t,T,T+\varepsilon) = -\frac{\log p(t,T+\varepsilon) - \log p(t,T)}{\varepsilon}.$$

2. The **spot rate**, $R(T, T+\varepsilon)$, for the period $[T, T+\varepsilon]$ is defined as

$$R(T,T+\varepsilon) = R(T,T,T+\varepsilon).$$

3. The **instantaneous forward rate** with maturity T, contracted at t is defined by

$$f(t,T) = \lim_{\varepsilon \to 0} R(t,T,T+\varepsilon) = -\frac{\partial \log p(t,T)}{\partial T}.$$

4. The **instantaneous short rate** at time t is defined by

$$r(t) = f(t,t).$$

We thus see that the bond market is different from any other market that we have considered so far, in the sense that the bond market contains an infinite number of assets (one bond type for each time of maturity). The basic goal in interest rate theory is roughly that of investigating the relations between all these different bonds. Somewhat more precisely we may pose the following general problems, to be studied below.

- What is a reasonable model for the bond market above?
- Is it possible to derive arbitrage free bond prices from a specification of the dynamics of the short rate of interest?
- Given a model for the bond market, how do you compute prices of interest rate derivatives , such as a European call option on an underlying bond ?

3.2 Bond pricing and martingale measures

We now go on to define the money account process B and introduce martingale measures into the bond market to model bond price.

Definition 3.3 The **money account** process is defined by

$$B(t) = \exp\Big\{\int_0^t r(u)\ du\Big\},$$

i.e.

$$\begin{cases} dB(t) &= r(t)B(t)\ dt, \\ B(0) &= 1. \end{cases}$$

where $\{r(t) : t \geq 0\}$ is an interest rate process.

Definition 3.4 **(Risk-neutral measure)** A *risk-neutral measure* (sometimes called a *martingale measure*) is any probability measure, equivalent to the market measure P, which makes all discounted asset prices martingales.

In what follows, we model the interest rate and bond prices processes in a generalised Black-Scholes framework. That is we assume that $W(t)$ is a standard Brownian motion and the filtration $\mathcal{F}(t)$ is the augmentation of the filtration generated by $W(t)$. The dynamics of the various processes are given as follows:

Short rate dynamics:

$$dr(t) = a(t,T)dt + b(t,T)dW(t), \tag{3.1}$$

Forward rate dynamics:

$$df(t,T) = \alpha(t,T)dt + \beta(t,T)dW(t). \tag{3.2}$$

Bond price dynamics:

$$dp(t,T) = p(t,T)\eta(t,T)dt + p(t,T)\varphi(t,T)dW(t), \tag{3.3}$$

We assume that in the above formula the coefficients meet standard conditions required to guarantee the existence of the various processes - that is, existence of solutions of the various stochastic differential equations. Therefore for every fixed T, discounted bond prices is

$$\begin{aligned} d\Big(\frac{p(t,T)}{B(t)}\Big) &= p(t,T)\ d\Big(\frac{1}{B(t)}\Big) + \frac{1}{B(t)}\ dp(t,T) \\ &= \Big[\eta(t,T) - r(t)\Big]\frac{p(t,T)}{B(t)}\ dt + \varphi(t,T)\frac{p(t,T)}{B(t)}\ dW(t), \end{aligned}$$

so P is a risk-neutral measure if and only if $\eta(t,T)$, the mean rate of return of $p(t,T)$ under P, is the interest rate $r(t)$. If the mean rate of return of $p(t,T)$ under P is not $r(t)$ at each time t and for each maturity T, we should change to a risk-neutral measure Q under which the mean rate of return is $r(t)$. In order to change measure, we take

$$\lambda(t,T) = \frac{\eta(t,T) - r(t)}{\varphi(t,T)}.$$

Then

$$d\Big(\frac{p(t,T)}{B(t)}\Big) = \frac{p(t,T)}{B(t)}\varphi(t,T)\Big(\lambda(t,T)\ dt + \ dW(t)\Big).$$

By Cameron-Martin-Girsanov Theorem, there exists a measure Q such that the process

$$\tilde{W}(t) = W(t) + \int_0^t \lambda(s)\ ds$$

is a Q Brownian motion. Since

$$\begin{aligned} E_Q\Big[d\big(\frac{p(t,T)}{B(t)}\big)\ \Big|\ \mathcal{F}(t)\Big] &= E_Q\Big[\frac{p(t,T)}{B(t)}\varphi(t,T)\ d\tilde{W}(t)\ \Big|\ \mathcal{F}(t)\Big] \\ &= \frac{p(t,T)}{B(t)}\varphi(t,T)E_Q\Big[d\tilde{W}(t)\ \Big|\ \mathcal{F}(t)\Big] \\ &= 0, \end{aligned}$$

then Q is a risk-neutral measure.

Remark 3.1 If we interpret η as the growth rate of the tradable, r as the growth rate of the riskless bond and φ as a measure of the risk of the asset, then

$$\lambda(t,T) = \frac{\eta(t,T) - r(t)}{\varphi(t,T)},$$

is the rate of extra return (above the risk-free rate) per unit of risk. As such it is often called the *market price of risk*.

Lemma 3.1 *Consider a fixed T-bond, and that Q is a risk neutral martingale measure. Then the price process for the T-bond is given by*

$$p(t,T) = E_Q\Big[\exp\Big\{-\int_t^T r(u)\ du\Big\}\ \Big|\ \mathcal{F}(t)\Big], \quad 0 \le t \le T.$$

where $\{r(t) : 0 \le t \le T\}$ is an interest rate process.

Proof. Under the risk neutral martingale measure Q, discounted bond prices is a martingale, then

$$\frac{p(t,T)}{B(t)} = E_Q\Big[\frac{p(T,T)}{B(T)}\ \Big|\ \mathcal{F}(t)\Big].$$

Thus

$$\begin{aligned} p(t,T) &= B(t)E_Q\Big[\frac{1}{B(T)}\ \Big|\ \mathcal{F}(t)\Big] \\ &= E_Q\Big[\exp\Big\{-\int_t^T r(u)\ du\Big\}\ \Big|\ \mathcal{F}(t)\Big],\quad 0 \le t \le T\ . \end{aligned}$$

■

3.3 The Cox-Ingersoll-Ross Model

In this section we turn to the problem of how to model an arbitrage free family of zero coupon bond price processes $\{p(\cdot, T) : T \geq 0\}$.

By Lemma 3.1, the price, $p(t,T)$, depends upon the behavior of the short rate of interest over the interval $[t,T]$, then a natural starting point is to give the dynamics of the short rate of interest. Let us model the short rate, under a fixed martingale measure Q, as the solution of the equation (3.1)

$$dr(t) = a(t, r(t))dt + b(t, r(t))dW(t),$$

where W is a Q Brownian motion.

Examples of short rate models include the following.

1. *Vasicek model* : $dr(t) = k(\mu - r(t))dt + \gamma dW(t)$, where k, μ, γ are constants;
2. *Cox-Ingersoll-Ross (CIR) model* : $dr(t) = k(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t)$, where k, μ, σ are constants;
3. *Ho-Lee model* : $dr(t) = \Phi(t)dt + \gamma dW(t)$, where $\Phi(t) = \dfrac{\mu(t)}{k(t)}$ and γ is a constant.
4. *Hull-White (extended Vasicek) model* : $dr(t) = k(t)(\mu(t) - r(t))dt + \gamma(t)dW(t)$;
5. *Hull-White (extended CIR) model* : $dr(t) = k(t)(\mu(t) - r(t))dt + \sigma(t)\sqrt{r(t)}dW(t)$.

We have noted that in the Vasicek and Ho-Lee models for $r(t)$, because $r(t)$ is Gaussian, there is a positive probability that $r(t) < 0$. The Cox-Ingersoll-Ross (CIR) model for $r(t)$ provides a stochastic differential equation for $r(t)$, the solution of which is always nonnegative.

Suppose we have n independent Brownian motions $W_1(t), W_2(t), \cdots, W_n(t)$ on a probability space $(\Omega, \mathcal{F}, P)$ and n Orstein-Uhlenbeck processes $X_1(t), X_2(t), \cdots, X_n(t)$ given by equations

$$dX_j(t) = -\frac{1}{2}kX_j(t)\ dt + \frac{1}{2}\sigma\ dW_j(t), \quad j = 1, 2, \cdots, n, \tag{3.4}$$

where $k > 0$ and $\sigma > 0$ are constants and $X_j(0) \in \mathbb{R}$ are given.
We take

$$\widetilde{X_j}(t) = e^{\frac{1}{2}kt}\ X_j(t). \qquad (\ \widetilde{X_j}(0) = X_j(0)\ )$$

Note

$$\begin{aligned} d\widetilde{X_j}(t) &= \frac{\partial \widetilde{X_j}(t)}{\partial t}\ dt + \frac{\partial \widetilde{X_j}(t)}{\partial X_j(t)}\ dX_j(t) + \frac{1}{2!}\frac{\partial^2 \widetilde{X_j}(t)}{\partial X_j^2(t)}\ (dX_j(t))^2 \\ &= \frac{1}{2}ke^{\frac{1}{2}kt}X_j(t)\ dt + e^{\frac{1}{2}kt}\ dX_j(t) \\ &= \frac{1}{2}\sigma e^{\frac{1}{2}kt}\ dW_j(t). \end{aligned}$$

Then

$$\widetilde{X_j}(t) - \widetilde{X_j}(0) = \int_0^t \frac{1}{2}\sigma e^{\frac{1}{2}ku}\ dW_j(u)$$

implies

$$X_j(t) = e^{-\frac{1}{2}kt}\Big[X_j(0) + \frac{1}{2}\ \sigma\ \int_0^t e^{\frac{1}{2}ku}\ dW_j(u)\Big]. \tag{3.5}$$

Since

$$\int_0^t e^{\frac{1}{2}ku}\ dW_j(u)$$

is a Itô integral, then, by Remark 2.2, $X_j(t)$ is normal with its mean function

$$m_j(t) = e^{-\frac{1}{2}kt}X_j(0) \tag{3.6}$$

and its covariance function

$$\begin{aligned} \rho(s,t) &= \rho\big(X_j(s), X_j(t)\big) \\ &= e^{-\frac{1}{2}ks}e^{-\frac{1}{2}kt}\rho\big(\widetilde{X_j}(s), \widetilde{X_j}(t)\big) \\ &= e^{-\frac{1}{2}k(s+t)}\int_0^{s\wedge t}\big(\frac{1}{2}\sigma e^{\frac{1}{2}ku}\big)^2\ du \\ &= \frac{1}{4}\sigma^2 e^{-\frac{1}{2}k(s+t)}\int_0^{s\wedge t} e^{ku}\ du. \end{aligned}$$

Thus its variance function is

$$\begin{aligned} \rho(t,t) &= \frac{\sigma^2}{4} e^{-kt} \int_0^t e^{ku}\, du \\ &= \frac{\sigma^2}{4} e^{-kt} \Big[\frac{1}{k} e^{ku} \Big|_0^t \Big] \\ &= \frac{\sigma^2}{4k} \big[1 - e^{-kt} \big]. \end{aligned} \tag{3.7}$$

Consider the process

$$r(t) := X_1^2(t) + X_2^2(t) + \cdots + X_n^2(t) \geq 0.$$

If $n = 1$, we have $r(t) = X_1^2(t)$ and for each t,

$$P\{r(t) > 0\} = 1,$$

but(see Fig. 3.1)

$$P\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\} = 1.$$

If $n \geq 2$,(see Fig. 3.2)

$$P\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} = 0.$$

Let $f(x_1, x_2, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$. Then

$$f_{x_j} = 2\ x_j$$

and

$$f_{x_i x_j} = \begin{cases} 2, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, 2, \cdots, n.$$

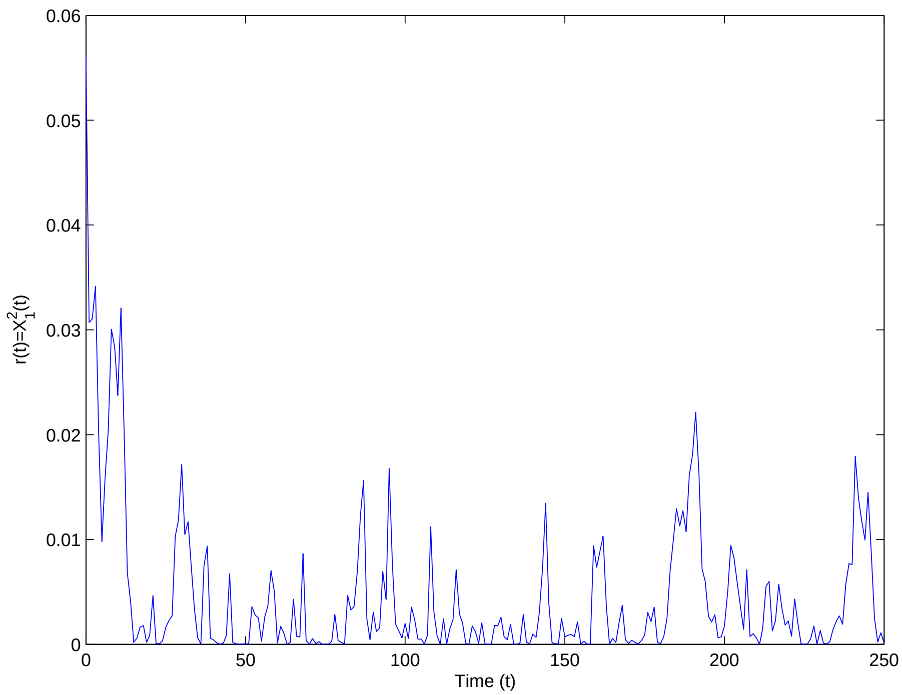

Figure 3.1: If $k = 0.205714, \sigma = 0.055855$, and $X_1(0) = 0.2369$, then $r(t)$ is a process above.

Thus, by Itô's formula,

$$\begin{aligned} dr(t) &= \left(\sum_{j=1}^{n} \frac{\partial r(t)}{\partial X_j(t)}\ dX_j(t)\right) + \frac{1}{2} \sum_{1\le i,j\le n} \frac{\partial^2 r(t)}{\partial X_i(t)\ \partial X_j(t)}\ dX_i(t)\ dX_j(t) \\ &= \sum_{j=1}^{n} 2X_j(t)\Big[-\frac{1}{2}kX_j(t)\ dt + \frac{1}{2}\sigma\ dW_j(t)\Big] + \frac{1}{2}\sum_{1\le j\le n} 2\ dX_j(t)\ dX_j(t) \\ &= -k\sum_{j=1}^{n} X_j^2(t)\ dt + \sigma \sum_{j=1}^{n} X_j(t)\ dW_j(t) + n\Big(\frac{1}{4}\sigma^2\ dt\Big) \\ &= k\Big(\frac{n\sigma^2}{4k} - r(t)\Big)dt + \sigma\sqrt{r(t)}\ \sum_{j=1}^{n} \frac{X_j(t)}{\sqrt{r(t)}}\ dW_j(t). \end{aligned}$$

Define

$$W(t) = \sum_{j=1}^{n} \int_0^t \frac{X_j(u)}{\sqrt{r(u)}}\ dW_j(u).$$

Since

$$\int_0^t \frac{X_j(u)}{\sqrt{r(u)}}\ dW_j(u), \quad j = 1, 2, \cdots, n,$$

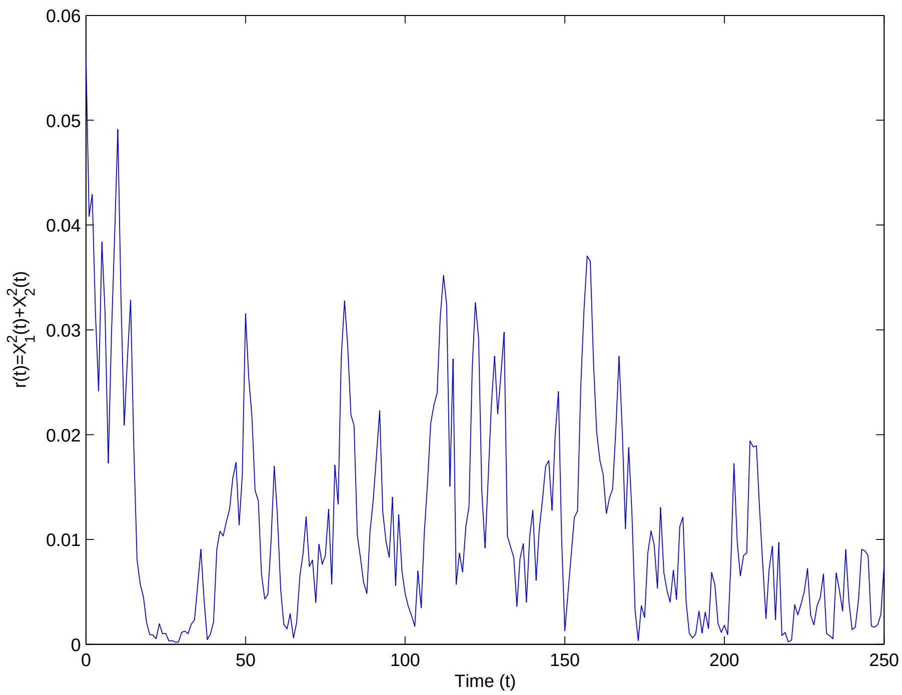

Figure 3.2: If $k = 0.205714, \sigma = 0.055855$, and $X_1(0) = 0.1675, X_2(0) = 0.1675$, then $r(t)$ is a process above.

are Itô integrals, then $W(t)$ is a martingale and

$$\begin{aligned} dW(t) &= \sum_{j=1}^{n} \frac{X_j(t)}{\sqrt{r(t)}}\, dW_j(t), \\ \big(dW(t)\big)^2 &= \sum_{j=1}^{n} \frac{X_j^2(t)}{r(t)}\, dt = \frac{r(t)}{r(t)}\, dt = dt. \end{aligned}$$

Thus $W(t)$ is a Brownian motion and we have

$$dr(t) = k\Big(\frac{n\sigma^2}{4k} - r(t)\Big)dt + \sigma\sqrt{r(t)}\, dW(t).$$

Definition 3.5 A *Cox-Ingersoll-Ross (CIR) process* is the process defined by an equation of the form

$$dr(t) = k\Big(\mu - r(t)\Big)dt + \sigma\sqrt{r(t)}\, dW(t).$$

where $k > 0$, $\mu > 0$, and $\sigma > 0$ are constants.

Take

$$n = \frac{4k\mu}{\sigma^2} > 0.$$

When n happens to be an integer, we have the representation

$$r(t) = \sum_{j=1}^{n} X_j^2(t) \geq 0,$$

but we do not require n to be an integer.

If $n < 2$ (i.e. $\mu < \frac{\sigma^2}{2k}$), then

$$P\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\} = 1.$$

This is not a good parameter choice.

If $n \geq 2$ (i.e. $\mu \geq \frac{\sigma^2}{2k}$), then

$$P\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} = 0.$$

With the CIR process, one can derive formulas under the assumption that $n = \frac{4k\mu}{\sigma^2}$ is a positive integer. Thus n Orstein-Uhlenbeck processes construct the process $r(t)$ which is nonnegative.

Suppose here is the distribution of $r(t)$ for fixed $t > 0$. Let $r(0) > 0$ be given. Then

$$X_1^2(0) + X_2^2(0) + \cdots + X_n^2(0) = r(0).$$

Since $X_j(t), j = 1, 2, \cdots, n$, is normal with mean function $m_j(t)$ (3.6) and variance function $\rho(t,t)$ (3.7), then

$$X_j(t) = m_j(t) + \sqrt{\rho(t,t)} Z_j,$$

where $Z_j \overset{i.i.d.}{\sim} N(0,1)$, $j = 1, 2, \cdots, n$.

Thus

$$\Big(\frac{X_j(t)}{\sqrt{\rho(t,t)}}\Big)^2 = \Big(\frac{m_j(t)}{\sqrt{\rho(t,t)}} + Z_j\Big)^2 \overset{i.i.d.}{\sim} \chi^2_{1,\delta_j},$$

where $j = 1, 2, \cdots, n$, χ^2_{1,δ_j} is a non-central chi-square distribution with 1 degree of freedom (d.f.) and noncentral parameter δ_j and

$$\begin{aligned} \delta_j &= \frac{m_j^2(t)}{\rho(t,t)} \\ &= \frac{4kX_j^2(0)}{\sigma^2(e^{kt} - 1)}. \end{aligned}$$

Since

$$r(t) = \rho(t,t)\sum_{j=1}^{n}\Big(\frac{X_j(t)}{\sqrt{\rho(t,t)}}\Big)^2,$$

then the distribution of $r(t)$ is $\rho(t,t) = \dfrac{\sigma^2}{4k}(1-e^{-kt})$ times a non-central chi-square with $n = \dfrac{4k\mu}{\sigma^2}$ degrees of freedom and non-central parameter δ, where $\delta = \sum_{j=1}^{n}\delta_j$.
Howevrer, consider the chi-square density having n degrees of freedom and non-central parameter δ, given by

$$f_{\chi^2_{n,\delta}}\big(y;\delta\big) = \frac{e^{-\frac{\delta}{2}}}{2\delta^{\frac{n}{4}-\frac{1}{2}}}\; e^{-\frac{y}{2}}\; y^{\frac{n}{4}-\frac{1}{2}}\; I_{\frac{n}{2}-1}\big(\sqrt{\delta y}\;\big), \tag{3.8}$$

where I_ν is the modifired Bessel function of order ν, given by

$$I_\nu(x) = \Big(\frac{x}{2}\Big)^2\; \sum_{j=0}^{\infty}\; \frac{(\frac{x}{2})^{2j}}{j!\;\Gamma(\nu+j+1)}.$$

Take

$$r = ny = \frac{\sigma^2}{4k}(1-e^{-kt})y \geq 0.$$

Thus the density of $r(t)$ is

$$\begin{aligned} g_{\chi^2_{n,\delta}}\big(r;\delta\big) &= \frac{e^{-\frac{\delta}{2}}}{2\;\delta^{\frac{n}{4}-\frac{1}{2}}}\; e^{-\frac{r}{2n}}\; \Big(\frac{r}{n}\Big)^{\frac{n}{4}-\frac{1}{2}}\; I_{\frac{n}{2}-1}\Big(\sqrt{\delta\frac{r}{n}}\;\Big)\;\Big|\frac{dy}{dr}\Big| \\ &= \frac{e^{-\frac{\delta}{2}}}{2\;n\;\delta^{\frac{n}{4}-\frac{1}{2}}}\; e^{-\frac{r}{2n}}\; \Big(\frac{r}{n}\Big)^{\frac{n}{4}-\frac{1}{2}}\; I_{\frac{n}{2}-1}\Big(\sqrt{\delta\frac{r}{n}}\;\Big) \end{aligned} \tag{3.9}$$

Remark 3.2 As $t\to\infty$, $m_j(t)\to 0$ and $\rho(t,t) = \frac{\sigma^2}{4k}$. We have

$$r(t) = \rho(t,t)\sum_{j=1}^{n}\Big(\frac{X_j(t)}{\sqrt{\rho(t,t)}}\Big)^2.$$

Then the limiting distribution of $r(t)$ is $\frac{\sigma^2}{4k}$ times a chi-square with $n = \frac{4k\mu}{\sigma^2}$ degrees of freedom (d.f.).
Note the chi-square density with $\frac{4k\mu}{\sigma^2}$ d.f. is

$$g(y) = \frac{1}{\Gamma\Big(\frac{2k\mu}{\sigma^2}\Big)\;2^{\frac{2k\mu}{\sigma^2}}}\; y^{\frac{2k\mu-\sigma^2}{\sigma^2}}\; e^{-\frac{y^2}{2}}.$$

Take $r = \frac{\sigma^2}{4k} y$. The limit density for $r(t)$ is

$$\begin{aligned} h(r) &= \frac{1}{\Gamma\left(\frac{2k\mu}{\sigma^2}\right) 2^{\frac{2k\mu}{\sigma^2}}} \left(\frac{4k}{\sigma^2}\, r\right)^{\frac{2k\mu-\sigma^2}{\sigma^2}} e^{-\frac{2k}{\sigma^2} r} \left|\frac{dy}{dr}\right| \\ &= \left(\frac{2k}{\sigma^2}\right)^{\frac{2k\mu}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2k\mu}{\sigma^2}\right)} r^{\frac{2k\mu}{\sigma^2}-1} e^{-\frac{2k}{\sigma^2} r}. \end{aligned}$$

This is the Gamma density parameters $\frac{2k\mu}{\sigma^2}$ and $\frac{\sigma^2}{2k}$ and is denoted by $G(\frac{2k\mu}{\sigma^2}, \frac{\sigma^2}{2k})$. Then the mean and variance of $r(t)$ are

$$\begin{aligned} E(r(t)) &= \frac{2k\mu}{\sigma^2} \cdot \frac{\sigma^2}{2k} = \mu, \\ Var(r(t)) &= \frac{2k\mu}{\sigma^2} \cdot \left(\frac{\sigma^2}{2k}\right)^2 = \frac{\mu\sigma^2}{2k}. \end{aligned}$$

3.4 Bond prices in the CIR model

The interest rate process $r(t)$ is given by

$$dr(t) = k\big(\mu - r(t)\big)dt + \sigma\sqrt{r(t)}\, dW(t),$$

where $r(0)$ is given. The bond price process is

$$p(t,T) = E\Big[e^{-\int_t^T r(u)\ du} \ \Big|\ \mathcal{F}(t)\Big].$$

Note

$$e^{-\int_0^t r(u)\ du} \cdot p(t,T) = E\Big[e^{-\int_0^T r(u)\ du} \ \Big|\ \mathcal{F}(t)\Big].$$

Then $e^{-\int_0^t r(u)\ du} \cdot p(t,T)$ is a martingale. Since $p(t,T)$ is random only through a dependence on $r(t)$, then we take the process $p(t,T)$ is the function $p(r(t),t,T)$ evaluated at $r(t)$ and denoted $p := p(r(t),t,T)$.

Note

$$\begin{aligned} & d\Big(e^{-\int_0^t r(u)\ du} \cdot p\Big) \\ &= e^{-\int_0^t r(u)\ du}\Big\{\big[-r(t)p + p_t\big]dt + p_r\ dr(t) + \frac{1}{2!}p_{rr}\ \big(dr(t)\big)^2\Big\} \\ &= e^{-\int_0^t r(u)\ du}\Big\{\big[-r(t)p + p_t + p_r k(\mu - r(t)) + \frac{1}{2}p_{rr}\sigma^2 r(t)\big]dt + p_r\sigma\sqrt{r(t)}\ dW(t)\Big\}. \end{aligned}$$

Since $e^{-\int_0^t r(u)\ du} \cdot p$ is a martingale, then

$$E\Big[d(e^{-\int_0^t r(u)\ du} \cdot p)\ \Big|\ \mathcal{F}(t)\Big] = 0$$

implies

$$e^{-\int_0^t r(u)\ du}\Big[-r(t)\ p + p_t + p_r k\big(\mu - r(t)\big) + \frac{1}{2} p_{rr}\sigma^2 r(t)\Big]dt$$

$$+e^{-\int_0^t r(u)\ du} p_r \sigma\sqrt{r(t)} E\Big(dW(t)\ \Big|\ \mathcal{F}(t)\Big) = 0.$$

By $E\big(dW(t)\ \big|\ \mathcal{F}(t)\big) = 0$, we obtain the partial differential equation

$$-r(t)p + p_t + p_r k\big(\mu - r(t)\big) + \frac{1}{2} p_{rr}\sigma^2 r(t) = 0, \tag{3.10}$$

where $0 \le t \le T$, $r(t) \ge 0$.

If we take the terminal condition that is

$$p(r(t), T, T) = 1, \quad r(T) \ge 0,$$

then this equation has a closed form solution.

We look for a solution of the form (Affine Term Structure),

$$p(r(t), t, T) = e^{-r(t)C(t,T) - A(t,T)}, \tag{3.11}$$

where $C(T,T) = 0$, $A(T,T) = 0$.

Note

$$\begin{aligned} p_t &= \big(-r(t)C_t(t,T) - A_t(t,T)\big)p, \\ p_r &= -C(t,T)p, \\ p_{rr} &= C^2(t,T)p. \end{aligned}$$

Then the equation (3.10) is

$$-r(t)p + p\big[-r(t)C_t(t,T) - A_t(t,T)\big] - C(t,T)p\ k\big(\mu - r(t)\big)$$

$$+\frac{1}{2}C^2(t,T)p\ \sigma^2 r(t) = 0$$

implies

$$-r(t)p\big[-1 - C_t(t,T) + kC(t,T) + \frac{1}{2}\sigma^2 C^2(t,T)\big] + p\big[-A_t(t,T) - \mu k C(t,T)\big] = 0.$$

Since $p > 0$, then for each $r(t)$

$$\begin{aligned} &-1 - C_t(t,T) + kC(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) = 0, \quad \cdots\cdots \text{ Riccati equation} \\ &A_t(t,T) + \mu k C(t,T) = 0, \end{aligned} \tag{3.12}$$

where $C(T,T) = 0$, and $A(T,T) = 0$.

In order to solute $C(t,T)$, we introduce another dependent variable $U(t,T)$ such that

$$\begin{aligned} C(t,T) &= \frac{U_t(t,T)}{\frac{1}{2}\sigma^2 U(t,T)} \\ &= 2U_t(t,T)\Big(\sigma^2 U(t,T)\Big)^{-1}. \end{aligned} \tag{3.13}$$

where $U(t,T) \neq 0$, for all t.

Differentiating $C(t,T)$ with respect to t we get

$$C_t(t,T) = 2U_{tt}(t,T)\Big(\sigma^2 U(t,T)\Big)^{-1} - 2U_t(t,T)\Big(\sigma^2 U(t,T)\Big)^{-2}\Big(\sigma^2 U_t(t,T)\Big) \tag{3.14}$$

Substituting $C(t,T)$ and $C_t(t,T)$ into (3.12),

$$\begin{aligned} &-1 - \Big[2U_{tt}(t,T)\Big(\sigma^2 U(t,T)\Big)^{-1} - 2U_t(t,T)\Big(\sigma^2 U(t,T)\Big)^{-2}\Big(\sigma^2 U_t(t,T)\Big)\Big] \\ &+k\Big[2U_t(t,T)\Big(\sigma^2 U(t,T)\Big)^{-1}\Big] + \frac{1}{2}\sigma^2\Big[2U_t(t,T)\Big(\sigma^2 U(t,T)\Big)^{-1}\Big]^2 = 0 \end{aligned}$$

implies

$$-1 - 2\Big(\sigma^2 U(t,T)\Big)^{-1} U_{tt}(t,T) + 2kU_t(t,T)\Big(\sigma^2 U(t,T)\Big)^{-1} = 0 \tag{3.15}$$

Multiplying both sides of the the equation (3.15) by $\sigma^2 U(t,T)$, we have

$$2U_{tt}(t,T) - 2kU_t(t,T) + \sigma^2 U(t,T) = 0. \tag{3.16}$$

The auxiliary equation is given by

$$(2D^2 - 2kD + \sigma^2)U(t,T) = 0, \tag{3.17}$$

where $D = \dfrac{d}{dt}$.

The solution to this quadratic in D is given by

$$\begin{aligned} D_1 &= \frac{k + \sqrt{k^2 + 2\sigma^2}}{2}, \\ D_2 &= \frac{k - \sqrt{k^2 + 2\sigma^2}}{2}. \end{aligned} \tag{3.18}$$

The general solution to the differential equation (3.16) will be given by

$$U(t,T) = ae^{D_1 t} + be^{D_2 t}, \tag{3.19}$$

where D_1 and D_2 are two values of D and a, b are constants.

Since the time starts at t and ends at T, then we rewrite the equation (3.19) as

$$U(t,T) = a^* e^{-D_1(T-t)} + b^* e^{-D_2(T-t)}, \tag{3.20}$$

where $a^* = ae^{D_1 T}$, $b^* = be^{D_1 T}$ are also constants.

Take $\tau = \frac{1}{2}\sqrt{k^2 + 2\sigma^2}$. Then

$$U(t,T) = a^* e^{-(\frac{k}{2}+\tau)(T-t)} + b^* e^{-(\frac{k}{2}-\tau)(T-t)}. \tag{3.21}$$

Note the derivative of $U(t,T)$ with respect to t will be given by

$$U_t(t,T) = -a^* e^{-(\frac{k}{2}+\tau)(T-t)}(\frac{k}{2}+\tau) - b^* e^{-(\frac{k}{2}-\tau)(T-t)}(\frac{k}{2}-\tau). \tag{3.22}$$

Substituting $U(t,T)$ and $U_t(t,T)$ into the equation (3.13), we get

$$\begin{aligned} C(t,T) &= \left(\frac{2}{\sigma^2}\right) \frac{-a^* e^{-(\frac{k}{2}+\tau)(T-t)}(\frac{k}{2}+\tau) - b^* e^{-(\frac{k}{2}-\tau)(T-t)}(\frac{k}{2}-\tau)}{a^* e^{-(\frac{k}{2}+\tau)(T-t)} + b^* e^{-(\frac{k}{2}-\tau)(T-t)}}, \\ &= \left(\frac{-2}{\sigma^2}\right) \frac{a^* e^{-\tau(T-t)}(\tau+\frac{k}{2}) - b^* e^{\tau(T-t)}(\tau-\frac{k}{2})}{a^* e^{-\tau(T-t)} + b^* e^{\tau(T-t)}}. \end{aligned} \tag{3.23}$$

Let

$$\begin{aligned} a^* &= \tau - \frac{k}{2}, \\ b^* &= \tau + \frac{k}{2}. \end{aligned}$$

Then

$$C(t,T) = \frac{e^{\tau(T-t)} - e^{-\tau(T-t)}}{(\tau+\frac{k}{2})e^{\tau(T-t)} + (\tau-\frac{k}{2})e^{-\tau(T-t)}}. \tag{3.24}$$

Take

$$A(t,T) = -\frac{2k\mu}{\sigma^2} \ln\left[\frac{2\tau e^{\frac{k}{2}(T-t)}}{(\tau+\frac{k}{2})e^{\tau(T-t)} + (\tau-\frac{k}{2})e^{-\tau(T-t)}}\right]. \tag{3.25}$$

Check

$$\begin{aligned}
A_t(t,T) &= -\frac{2k\mu}{\sigma^2}\frac{(\tau+\frac{k}{2})e^{\tau(T-t)}+(\tau-\frac{k}{2})e^{-\tau(T-t)}}{2\tau e^{\frac{k}{2}(T-t)}} \\
&\quad \frac{2\tau e^{\frac{k}{2}(T-t)}\Big[(\tau-\frac{k}{2})(\tau+\frac{k}{2})e^{\tau(T-t)}-(\tau+\frac{k}{2})(\tau-\frac{k}{2})e^{-\tau(T-t)}\Big]}{\Big[(\tau+\frac{k}{2})e^{\tau(T-t)}+(\tau-\frac{k}{2})e^{-\tau(T-t)}\Big]^2} \\
&= -k\mu\frac{e^{\tau(T-t)}-e^{-\tau(T-t)}}{(\tau+\frac{k}{2})e^{\tau(T-t)}+(\tau-\frac{k}{2})e^{-\tau(T-t)}} \\
&= -k\mu C(t,T).
\end{aligned} \tag{3.26}$$

Thus the solution (3.12) is given by

$$\begin{aligned}
C(t,T) &= \frac{sinh\big(\tau(T-t)\big)}{\tau cosh\big(\tau(T-t)\big)+\frac{1}{2}ksinh\big(\tau(T-t)\big)}, \\
A(t,T) &= -\frac{2\mu k}{\sigma^2}\log\Big[\frac{\tau e^{\frac{1}{2}k(T-t)}}{\tau cosh\big(\tau(T-t)\big)+\frac{1}{2}ksinh\big(\tau(T-t)\big)}\Big].
\end{aligned} \tag{3.27}$$

where $\tau=\frac{1}{2}\sqrt{k^2+2\sigma^2}$, $cosh(u)=\frac{e^u+e^{-u}}{2}$, $sinh(u)=\frac{e^u-e^{-u}}{2}$.

3.5 European Options on a bond

The value at time t of the European call option on a bond in the CIR model is

$$V\big(t,r(t)\big)=E\Big[e^{-\int_t^{T_1}r(u)\ du}\cdot\big(p(T_1,T_2)-K\big)^+\ \Big|\ \mathcal{F}(t)\Big], \tag{3.28}$$

where K is the expiration value, T_1 is the expiration time of the option, T_2 is the maturity time of the bond, $0\le t\le T_1\le T_2$, and $r(t)$ is a CIR process.
Let the terminal condition be given

$$V\big(T_1,r(T_1)\big)=\Big(p\big(r(T_1),T_1,T_2\big)-K\Big)^+,\quad r(T_1)>0.$$

As usual,

$$e^{-\int_0^t r(u)\ du}\cdot V\big(t,r(t)\big)=E\Big[e^{-\int_0^{T_1}r(u)\ du}\cdot\big(p(T_1,T_2)-K\big)^+\ \Big|\ \mathcal{F}(t)\Big].$$

Then $e^{-\int_0^t r(u)\ du} \cdot V(t, r(t))$ is a martingale, and we denote $V := V(r(t), t, T_1)$.
Note

$$\begin{aligned}
& d\Big(e^{-\int_0^t r(u)\ du} \cdot V\Big) \\
& = e^{-\int_0^t r(u)\ du}\Big\{\big[-r(t) \cdot V + V_t\big]dt + V_r\ dr(t) + \frac{1}{2!}V_{rr}\ (dr(t))^2\Big\} \\
& = e^{-\int_0^t r(u)\ du}\Big\{\big[-r(t)V + V_t + V_r k\big(\mu - r(t)\big) + \frac{1}{2}V_{rr}\sigma^2 r(t)\big]dt + V_r\sigma\sqrt{r(t)}\ dW(t)\Big\},
\end{aligned}$$

Since $e^{-\int_0^t r(u)\ du} \cdot V$ is a martingale, then

$$E\Big[d\big(e^{-\int_0^t r(u)\ du} \cdot V\big)\ \Big|\ \mathcal{F}(t)\Big] = 0$$

implies

$$\begin{aligned}
& e^{-\int_0^t r(u)\ du}\Big[-r(t)V + V_t + V_r k\big(\mu - r(t)\big) + \frac{1}{2}V_{rr}\ \sigma^2 r(t)\Big]dt \\
& + e^{-\int_0^t r(u)\ du} V_r\sigma\sqrt{r(t)}E\Big(dW(t)\ \Big|\ \mathcal{F}(t)\Big) = 0.
\end{aligned}$$

By $E\big(dW(t)\ \big|\ \mathcal{F}(t)\big) = 0$, we obtain the partial differential equation

$$-r(t)V\big(t, r(t)\big) + V_t\big(t, r(t)\big) + V_r\big(t, r(t)\big)k\big(\mu - r(t)\big) + \frac{1}{2}V_{rr}\big(t, r(t)\big)\sigma^2 r(t) = 0, \quad (3.29)$$

where $0 \le t \le T_1$, $r(t) \ge 0$.

Chapter 4

Numerical Computation

The object of interest rate modelling is often used to find the prices for derivatives. The chapter explores our numerical methods, explicit finite difference and Monte Carlo methods to pricing the bond option.

4.1 Valuing Bond Option Prices by the Explicit Finite Difference Method

Following Section 3.5, we know an European call option on bond in the CIR model, with price V, satisfies the equation,

$$-r(t)V\big(t,r(t)\big)+V_t\big(t,r(t)\big)+V_r\big(t,r(t)\big)k\big(\mu-r(t)\big)+\frac{1}{2}V_{rr}\big(t,r(t)\big)\sigma^2 r(t)=0, \qquad (4.1)$$

where $0\le t\le T_1$, T_1 is the expiration time of the option, and r satisfies

$$dr(t)=k\Big(\mu-r(t)\Big)dt+\sigma\sqrt{r(t)}\,dW(t), \quad k>0,\mu>0,\sigma>0 \text{ are constants.} \qquad (4.2)$$

Let the terminal condition be given

$$V\big(T_1,r(T_1)\big)=\Big(p\big(r(T_1),T_1,T_2\big)-K\Big)^+,$$

where T_2 is the maturity time of the bond and $r(T_1)\ge 0$.

4.1.1 The Explicit Finite Difference Method

Let the parameters $r_{\min}$ and $r_{\max}$ be the smallest and largest values of r considered by the model, and t_0 is the current time. Partition $[t_0, T_1]$ into I subintervals with fixed length

$$\Delta t = \frac{T_1 - t_0}{I}.$$

Also, the partition $[r_{\min}, r_{\max}]$ contain J intervals with a constant change in r

$$\Delta r = \frac{r_{\max} - r_{\min}}{J}.$$

Let

$$V_{i,j} = V(t_0 + i\Delta t, r_j),$$

where $r_j = r_{\min} + j\Delta r$, $i = 0, 1, \cdots, I$, and $j = 0, 1, \cdots, J$, that is, partition $(I+1) \times (J+1)$ points of $[t_o, T_1] \times [r_{\min}, r_{\max}]$, (see Fig. 4.1).

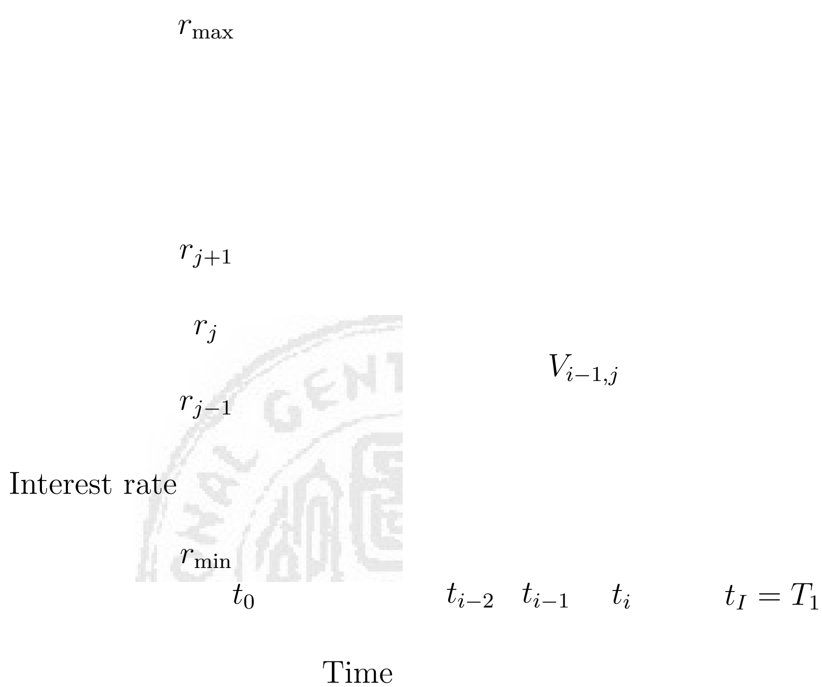

Figure 4.1: The grid of option prices.

By Taylor's polynomial,

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2!}f''(x)(\Delta x)^2 + \frac{1}{3!}f'''(x)(\Delta x)^3 + O(\Delta x^4),$$

and

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2!}f''(x)(\Delta x)^2 - \frac{1}{3!}f'''(x)(\Delta x)^3 + O(\Delta x^4).$$

Then $\frac{\partial V}{\partial r}$, $\frac{\partial^2 V}{\partial r^2}$, and $\frac{\partial V}{\partial t}$ with respect to r at node $(i-1, j)$ are approximated as follows,

$$\begin{aligned}
\frac{\partial V}{\partial r} &= \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta r} + O(\Delta r^2), \\
\frac{\partial^2 V}{\partial r^2} &= \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{\Delta r^2} + O(\Delta r^2), \qquad (4.3) \\
\frac{\partial V}{\partial t} &= \frac{V_{i,j} - V_{i-1,j}}{\Delta t} + O(\Delta t).
\end{aligned}$$

Substituting (4.3) into (4.1) gives

$$V_{i-1,j} = a_{j-1}V_{i,j-1} + a_j V_{i,j} + a_{j+1}V_{i,j+1} + O(\Delta t, \Delta r^2), \qquad (4.4)$$

where

$$\begin{aligned}
a_{j-1} &= \frac{1}{1+r_j\Delta t}\left[-\frac{k(\mu - r_j)\Delta t}{2\Delta r} + \frac{\sigma^2 r_j \Delta t}{2\Delta r^2}\right], \\
a_j &= \frac{1}{1+r_j\Delta t}\left[1 - \frac{\sigma^2 r_j \Delta t}{\Delta r^2}\right], \\
a_{j+1} &= \frac{1}{1+r_j\Delta t}\left[\frac{k(\mu - r_j)\Delta t}{2\Delta r} + \frac{\sigma^2 r_j \Delta t}{2\Delta r^2}\right].
\end{aligned}$$

Define

$$\begin{aligned}
q_{j,j-1} &= -\frac{k(\mu - r_j)\Delta t}{2\Delta r} + \frac{\sigma^2 r_j \Delta t}{2\Delta r^2}, \\
q_{j,j} &= 1 - \frac{\sigma^2 r_j \Delta t}{\Delta r^2}, \\
q_{j,j+1} &= \frac{k(\mu - r_j)\Delta t}{2\Delta r} + \frac{\sigma^2 r_j \Delta t}{2\Delta r^2},
\end{aligned}$$

so that equation (4.4) becomes

$$V_{i-1,j} = \frac{1}{1+r_j\Delta t}\left[q_{j,j-1}V_{i,j-1} + q_{j,j}V_{i,j} + q_{j,j+1}V_{i,j+1}\right] + O(\Delta t, \Delta r^2), \qquad (4.5)$$

where local truncation error is $O(\Delta t, \Delta r^2)$, and accumulated truncation error is $O(\Delta t^2, \Delta t \Delta r^2)$.

We can verify that it is always true that

$$q_{j,j-1} + q_{j,j} + q_{j,j+1} = 1. \tag{4.6}$$

If $q_{j,j-1}, q_{j,j}$, and $q_{j,j+1}$ are all positive, then $q_{j,j-1}, q_{j,j}$, and $q_{j,j+1}$ can be interpreted as probabilities of moving from r_j to r_{j-1}, r_j and r_{j+1}, respectively, during time interval Δt. Thus the explicit finite difference method is equivalent to a trinomial lattice approach. In Section 4.1.3, this equivalence is used to explain the conditions required to ensure convergence .

4.1.2 The Transformation of Variables

Generalizing from this, we can define a new state variable $\phi(r(t), t)$ that has a constant instantaneous standard deviation. Take

$$\phi(r(t), t) = \sqrt{r(t)}.$$

From Itô's lemma and equation (4.2), the process followed by $\phi(r(t), t)$ in a risk-neutral world is

$$d\phi(r(t), t) = \psi(r(t), t)dt + \frac{1}{2}\sigma dW(t), \tag{4.7}$$

where

$$\begin{aligned} \psi(r(t), t) &= \frac{k(\mu - r(t))}{2\sqrt{r(t)}} - \frac{\sigma^2}{8\sqrt{r(t)}} \\ &= \frac{4k\mu - \sigma^2}{8\phi(r(t), t)} - \frac{\phi(r(t), t)k}{2}. \end{aligned} \tag{4.8}$$

The state variable ϕ can be modeled in the same way as r. A grid is constructed for values of ϕ equal to $\phi_0, \phi_1, \cdots, \phi_n$, where ϕ_0 is the largest multiple of $\Delta\phi$ less than $\sqrt{r_{\min}}$, $\phi_j = \phi_0 + j\Delta\phi$, and n is the smallest integer such that $\phi_n \geq \sqrt{r_{\max}}$, and the equation in

(4.5) become

$$
\begin{aligned}
q_{j,j-1} &= -\psi_j \frac{\Delta t}{2\Delta\phi} + \sigma^2 \frac{\Delta t}{8\Delta\phi^2}, \\
q_{j,j} &= 1 - \sigma^2 \frac{\Delta t}{4\Delta\phi^2}, \\
q_{j,j+1} &= \psi_j \frac{\Delta t}{2\Delta\phi} + \sigma^2 \frac{\Delta t}{8\Delta\phi^2},
\end{aligned}
\tag{4.9}
$$

where

$$\psi_j = \frac{4k\mu - \sigma^2}{8\phi_j} - \frac{\phi_j k}{2}.$$

In the explicit finite difference method, every node ϕ_j is constant, then ψ_j is also constant. It has the simplifying property that $q_{j,j-1}, q_{j,j}$, and $q_{j,j+1}$ are independent of j.

4.1.3 The Conditions of Convergence and Stability

When using the explicit finite difference method, it is important to ensure that when Δt and $\Delta\phi \to 0$, the estimated value of the option converges to its true value .

Let $V(r(t), t)$ be the call option's true value, and $v(r(t), t)$ be the estimated one. Then, by equation (4.5), the estimated value v satisfies

$$v_{i-1,j} = \frac{1}{1 + r_j\Delta t}\left[q_{j,j-1}v_{i,j-1} + q_{j,j}v_{i,j} + q_{j,j+1}v_{i,j+1}\right]. \tag{4.10}$$

Thus the error of the explicit finite difference method is

$$\varepsilon_{i-1,j} = \frac{1}{1 + r_j\Delta t}\left[q_{j,j-1}\varepsilon_{i,j-1} + q_{j,j}\varepsilon_{i,j} + q_{j,j+1}\varepsilon_{i,j+1}\right] + O(\Delta t, \Delta r^2), \tag{4.11}$$

where $\varepsilon_{i-1,j} = V_{i-1,j} - v_{i-1,j}$ and $i = 1, 2, \cdots, I - 1$.

Since v agrees with V on the boundary,

$$
\begin{aligned}
&\varepsilon_{I,j} = 0, && j = 0, 1, \cdots, J, \\
&\varepsilon_{i,0} = \varepsilon_{i,J} = 0, && i = 0, 1, \cdots, I - 1.
\end{aligned}
\tag{4.12}
$$

Suppose $q_{j,j-1}, q_{j,j}$, and $q_{j,j+1}$ are nonnegative, that is, the following equation must hold,

$$\begin{aligned} \sigma^2 \frac{\Delta t}{4\Delta\phi^2} < 1, \\ \mid \psi_j \mid < \frac{\sigma^2}{4\Delta\phi}. \end{aligned} \tag{4.13}$$

Thus

$$\begin{aligned} |\varepsilon_{i-1,j}| &\leq q_{j,j-1}|\varepsilon_{i,j-1}| + q_{j,j}|\varepsilon_{i,j}| + q_{j,j+1}|\varepsilon_{i,j+1}| + a \cdot O(\Delta t^2, \Delta t \Delta r^2) \\ &\leq \|\varepsilon_j\| + a \cdot O(\Delta t^2, \Delta t \Delta r^2), \quad i = 1, 2, \cdot, I-1, \end{aligned} \tag{4.14}$$

where

$$a \in \mathbb{R}, \quad \|\varepsilon_j\| = \max_{i=0,\cdots,\ I} |\varepsilon_{i,j}|,$$

and $O(\Delta t^2, \Delta t \Delta r^2)$ is the accumulated truncation error.

By equation (4.14), we get

$$\|\varepsilon_{j+1}\| \leq \|\varepsilon_j\| + a \cdot O(\Delta t^2, \Delta t \Delta r^2), \tag{4.15}$$

and since $\|\varepsilon_0\| = 0$ we easily calculate that

$$\begin{aligned} \|\varepsilon_j\| &\leq a \cdot O(\Delta t^2, \Delta t \Delta r^2) \cdot j \\ &= a \cdot j\Delta t \cdot O(\Delta t, \Delta r^2) \\ &\leq a \cdot T \cdot O(\Delta t, \Delta r^2), \end{aligned} \tag{4.16}$$

as $j\Delta t \leq T$. When ψ_j is bounded, equation (4.13) is satisfied and then the convergence can be ensured.

4.1.4 The Modification of Branches

By equation (4.8), since ϕ can take on any positive value, ψ may or may not be bounded. It follows that the explicit finite difference method may diverge. However, the method can be modified to overcome this problem. Instead of insisting that we move from ϕ_j to one of ϕ_{j-1}, ϕ_j, and ϕ_{j+1} in time Δt, we allow a movement from ϕ_j to one of $\phi_{\ell-1}, \phi_\ell$, and $\phi_{\ell+1}$, where ℓ is not necessarily equal to j and ℓ is an integer. In Figure 4.2, (a)-(e) show the situations where $\ell = j, \ell = j+1, \ell = j-1, \ell < j-1$, and $\ell > j+1$, respectively.

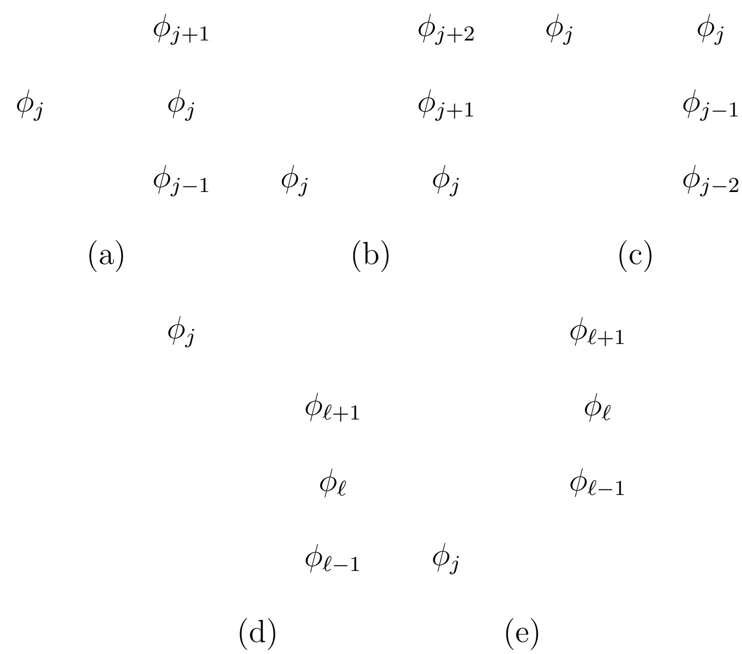

Figure 4.2: Alternative Branching Procedures.

In all cases, we choose ℓ so that ϕ_ℓ is the value of ϕ on the grid closest to $\phi_j + \psi_j \Delta t$. The probabilities of ϕ_j moving to $\phi_{\ell-1}, \phi_\ell$, and $\phi_{\ell+1}$ are chose to make the first and second moments of the change in ϕ in time interval Δt correct in the limit as $\Delta t \to 0$. The equations that must be satisfied are:

$$\begin{cases} q_{j,\ell-1}(\ell-1)\Delta\phi + q_{j,\ell}\ \ell\ \Delta\phi + q_{j,\ell+1}(\ell+1)\Delta\phi = E(\phi) \\ q_{j,\ell-1}(\ell-1)^2\Delta\phi^2 + q_{j,\ell}\ \ell^2\Delta\phi^2 + q_{j,\ell+1}(\ell+1)^2\Delta\phi^2 = \frac{\sigma^2}{4}\Delta t + E(\phi)^2, \\ q_{j,\ell-1} + q_{j,\ell} + q_{j,\ell+1} = 1, \end{cases} \tag{4.17}$$

where $E(\phi)$ is the expected value of $\phi - \phi_0$ at the end of the time interval, Δt. The solution to these equations is

$$\begin{aligned} q_{j,\ell-1} &= \frac{1}{2}\left[\ell^2+\ell-(1+2\ell)\frac{E(\phi)}{\Delta\phi}+\frac{E(\phi)^2}{\Delta\phi^2}+\frac{\sigma^2}{4}\frac{\Delta t}{\Delta\phi^2}\right], \\ q_{j,\ell} &= 1-\ell^2+2\ell\frac{E(\phi)}{\Delta\phi}-\frac{E(\phi)^2}{\Delta\phi^2}-\frac{\sigma^2}{4}\frac{\Delta t}{\Delta\phi^2}, \\ q_{j,\ell+1} &= \frac{1}{2}\left[\ell^2-\ell+(1-2\ell)\frac{E(\phi)}{\Delta\phi}+\frac{E(\phi)^2}{\Delta\phi^2}+\frac{\sigma^2}{4}\frac{\Delta t}{\Delta\phi^2}\right]. \end{aligned} \tag{4.18}$$

Recalling equation (4.8), define

$$\alpha_1=\frac{4k\mu-\sigma^2}{8}, \quad \alpha_2=\frac{k}{2}.$$

If ψ is small, an examination of the errors in the way in which the differential equation is approximated suggests that a sensible value for $(\frac{1}{2}\sigma)^2\Delta t/\Delta\phi^2 = 1/3$ (see [14]). We find that this works well. It is easy to show $\ell = j$ when

$$-\frac{1}{2}\leq\left[\frac{\alpha_1}{\phi}-\alpha_2\phi\right]\frac{\Delta t}{\Delta\phi}\leq\frac{1}{2}. \tag{4.19}$$

Assuming α_1 and α_2 are positive, this condition reduces to

$$\phi_{\min}\leq\phi\leq\phi_{\max}, \tag{4.20}$$

where equations is

$$\begin{aligned} \phi_{\min} &= \frac{-\beta+\sqrt{\beta^2+4\alpha_1\alpha_2}}{2\alpha_2}, \\ \phi_{\max} &= \frac{\beta+\sqrt{\beta^2+4\alpha_1\alpha_2}}{2\alpha_2}, \\ \beta &= \frac{\Delta\phi}{2\Delta t}. \end{aligned}$$

The values of ϕ considered on the grid for the explicit finite difference method are $\phi_0, \phi_1, \cdots, \phi_n$, where ϕ_0 is the largest multiple of $\Delta\phi$ less than $\phi_{\min}$, $\phi_j = \phi_0 + j\Delta\phi$, and n is the smallest integer such that $\phi_n \geq \phi_{\max}$. It is assumed that $\Delta\phi$ is also chosen so that some multiple of $\Delta\phi$ equals the current value of ϕ.

When $1 \leq j \leq n-1$, the explicit finite difference method (trinomial lattice) approach operates in the usual way. When the value ϕ_0 is reached, the three possible values of ϕ after a time interval Δt are ϕ_0, ϕ_1, and ϕ_2. The probabilities of moving to these values are calculated from equation (4.18), with $j = 0$ and $\ell = 1$. Similarly, when the value ϕ_n is reached, the three possible values of ϕ after a time interval Δt are ϕ_{n-2}, ϕ_{n-1}, and ϕ_n. The probabilities of moving to these values are calculated from equation (4.18), with $j = n$ and $\ell = n-1$. Since the short-term interest rate is ϕ^2, the value of the bond prior to maturity can be calculated using

$$V_{i,j} = \frac{1}{1+\phi_j^2 \Delta t}\left[q_{j,j-1}V_{i+1,j-1} + q_{j,j}V_{i+1,j} + q_{j,j+1}V_{i+1,j+1}\right], \tag{4.21}$$

for $j = 1, 2, \cdots, n-1$,

$$V_{i,0} = \frac{1}{1+\phi_0^2 \Delta t}\left[q_{0,0}V_{i+1,0} + q_{0,1}V_{i+1,1} + q_{0,2}V_{i+1,2}\right], \tag{4.22}$$

and

$$V_{i,n} = \frac{1}{1+\phi_n^2 \Delta t}\left[q_{n,n-2}V_{i+1,n-2} + q_{n,n-1}V_{i+1,n-1} + q_{n,n}V_{i+1,n}\right]. \tag{4.23}$$

4.2 Valuing Bond Option Prices by the Basic Monte Carlo Method

From equation 3.28, we know the value at time t of the European call option on a bond in the CIR model is

$$V\big(t, r(t)\big) = E\Big[e^{-\int_t^{T_1} r(u)\ du} \cdot \big(p(T_1, T_2) - K\big)^+ \ \Big|\ \mathcal{F}(t)\Big], \tag{4.24}$$

where K is the expiration value, T_1 is the expiration time of the option, T_2 is the maturity time of the bond, $0 \le t \le T_1 \le T_2$, r is a CIR process, and r satisfies

$$dr(t) = k\Big(\mu - r(t)\Big)dt + \sigma\sqrt{r(t)}\ dW(t), \quad k > 0, \mu > 0, \sigma > 0 \text{ are constants.} \tag{4.25}$$

Take $t = t_0$, and

$$H(T_1, \omega) = e^{-\int_{t_0}^{T_1} r(u)\ du} \cdot \big(p(T_1, T_2) - K\big)^+, \tag{4.26}$$

where the measure of r is P, and $\omega \in \Omega$ is a sample point in the path space $\Omega \equiv \Omega(r_1, \cdots, r_J)$ of the interest rate variables $r = (r_1, \cdots, r_J)$. The expectation is an integral over Ω:

$$V\big(t_0, r(t_0)\big) = \int_\Omega H(T_1, \omega)\ dP. \tag{4.27}$$

If r is a discrete time process, taking values only at times $t_0 < t_1 < \cdots < t_I = T_1$, the integral will become

$$V\big(t_0, r(t_0)\big) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H(T_1, r_1(t_0), \cdots, r_J(t_I)) g(r_1(t_0), \cdots, r_J(t_I))\, dr_1(t_0) \cdots dr_J(t_I) \tag{4.28}$$

where $r_j(t_i)$ is the value of the interest rate variable r_j at time t_i, for a density function g. This is a $J \times (I+1)$-dimensional integral. If (4.28) is to be a good approximation to (4.27), then I must be large and it is not easy to use a numerical integration method to compute the integral.

In this case a Monte Carlo integration method may be appropriate. The idea is to sample the joint density of $r_j(t_i)_{j=1,\cdots,J, i=t_0,\cdots,t_I}$ to obtain N sample points, $(r_j^{(n)}(t_0), \cdots, r_j^{(n)}(t_I))$, $n = 1, \cdots, N$, from the sample space Ω, $(r_1^{(n)}, \cdots, r_J^{(n)})$. A single sample point is a path of values of r. The value of the European call option on a bond in the CIR model is estimated to be

$$\widetilde{V}\big(t_0, r(t_0)\big) = \frac{1}{N} \sum_{n=1}^{N} H(r^{(n)}(t_0), \cdots, r^{(n)}(t_I)). \tag{4.29}$$

This is the Monte Carlo estimate of the value of the derivative.

Since the interest rate $r(t)$ is the only random variable, sampling the density is done by evolving r from their initial values. Starting from an initial value of $r(t_0)$ at time t_0, we use a discrete approximation for the equation (4.25) to evolve $r(t)$ up to T_1. The path we have found is a sample from the space of paths, Ω. $H(T_1, r(T_1))$ along this sample path is compute. The value of $H(T_1, r(T_1))$ will depend upon the random numbers generated in order to evolve $r(t)$, so the simulation must be repeated many times to expect to accurately reflect the distribution of $H(T_1, r(T_1))$. The sampled values of $H(T_1, r(T_1))$ are discounted back to time t_0, using the generated values of $r(t)$ as discount rates, and the arithmetic average of these discounted values is the Monte Carlo estimate of the value of the European call option on a bond in the CIR model.

The basic Monte Carlo method can be broken up into stages:

1. Divide the period $[t_0, T_1]$ into I times steps. Set $\Delta t = \frac{T_1 - t_0}{I}$.

2. Compute N sample paths $r^{(n)}(t), n = 1, \cdots, N$, for $t = t_0, t_0 + \Delta t, \cdots, T_1$. For $n = 1, \cdots, N$, set $r^{(n)}(t_0) = r(t_0)$, the value of r at time t_0. At each time step the next value of $r^{(n)}(t)$ is found from its discretised process. For instance, the Euler

discretisation of the equation (4.25) is

$$r^{(n)}(t+\Delta t) = r^{(n)}(t) + k\Big(\mu - r^{(n)}(t)\Big)\Delta t + \sigma\sqrt{r^{(n)}(t)}\sqrt{\Delta t}\; y^{(n)}(t+\Delta t), \qquad (4.30)$$

where $y^{(n)}(t+\Delta t) \overset{i.i.d.}{\sim} N(0,1), t = t_0, \cdots, T_1 - \Delta t$, is a sequence of independent standard normal variables.

3. Obtain values $r^{(n)}(T_1), n = 1, \cdots, N$, at time T_1.

4. Compute $H(T_1, r^{(n)}(T_1)), n = 1, \cdots, N$, and discount back to time t_0 using the generated values of the interest rate $r(t)$; that is, compute

$$V^{(n)}(t_0) = \exp\Big\{-\sum_{t=t_0+\Delta t}^{T_1} r^{(n)}(t)\Delta t\Big\} \cdot H\big(T_1, r^{(n)}(T_1)\big), \quad n = 1, \cdots, N. \qquad (4.31)$$

$V^{(n)}(t_0)$ is the value at time t_0 of the European call option on a bond in the CIR model along the n'th sample path.

5. The Monte Carlo estimate $\widetilde{V}(t_0)$ of the value $V(t_0)$ of the European call option on a bond in the CIR model at time t_0 is the average of the $V^{(n)}(t_0)$ that is,

$$\widetilde{V}(t_0) = \frac{1}{N}\sum_{n=1}^{N} V^{(n)}(t_0). \qquad (4.32)$$

4.3 Numerical Result

We will shows the results of using the procedure to value a 10-Year bond with face value of \$ 100.

Recalling equation (4.2), the CIR model is

$$dr(t) = k\Big(\mu - r(t)\Big)dt + \sigma\sqrt{r(t)}\; dW(t),$$

where $k > 0, \mu > 0$, and $\sigma > 0$ are constants.

The model exhibits mean reversion of the interest rate, causing the rate to be pulled downward when it is above the long run average rate and be pulled upward when it is below the long run average rate. The coefficient k is the speed of this mean reversion, μ is the long run average rate, and σ is the volatility. By the lest squares method, we take $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$ such that the CIR model is similar to the real interest rate (see Appendix A). Figure 4.3 shows the estimated interest rate value in the CIR model when the parameter given above and the real interest rate value.

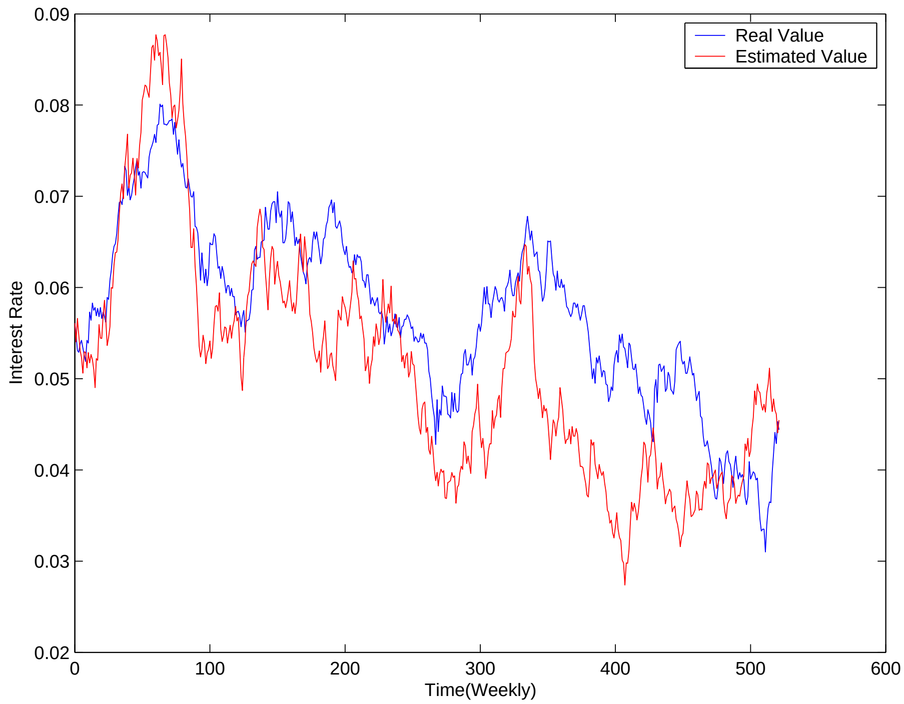

Figure 4.3: The estimated interest rate in CIR model for $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.

4.3.1 Numerical Result by the Explicit Finite Difference Method

By equations (3.10) and (4.1), we know the bond process and the process of the European call option on bond have the same partial differential equation. Thus using the explicit finite difference method in Section 4.1, we get Table 4.1 shows the values of the 10-Year zero coupon bond that maturity time is after 6 months in different current interest rate. And Table 4.2 shows the price of a 5-Year European call option which expiration time is 6 months early than the maturity time of the bond in different exercise price.

$100 face value	Current Short-Term Interest Rate					
	2%	4%	6%	8%	10%	12%
Bond Price	98.9140	97.9812	97.0572	96.1398	95.2279	94.3339
Analytic Solution	98.9127	97.9797	97.0555	96.1401	95.2332	94.3350

Table 4.1: The 10-Year bond prices that maturity time is after 6 months for the Cox, Ingersoll, and Ross model with $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.

Exercise Price	Current Short-Term Interest Rate					
	2%	4%	6%	8%	10%	12%
90	6.5468	5.9042	5.3088	4.7707	4.2754	3.8227
93	4.0200	3.5283	3.0761	2.6710	2.3018	1.9675
96	1.4936	1.1522	0.8434	0.5714	0.3281	0.1122

Table 4.2: Prices of a 5-Year call option on a 10-Year bond that maturity time is after 6 months for the the Cox, Ingersoll, and Ross model with $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.

4.3.2 Numerical Result by the Monte Carlo Method

As usual, we shows the results of using the basic Monte Carlo method to value a 5-Year European call option which expiration time is 6 months early than maturity time of 10-Year bond with face value of $ 100, and take the same parameter $k = 0.205714, \mu = 0.058856, \sigma = 0.055855$. We simulate below :

Step1 : We divide 10 years into 521 times steps, then $\Delta t = \frac{10}{521}$ (year) $\approx \frac{7}{365}$ (year) $= 1$ (week).

Step2 : Compute N sample paths of CIR interest rate process $r^{(n)}(t), n = 1, \cdots, N$, for $t = 0, \Delta t, \cdots, 521$ (week). For $n = 1, \cdots, N$ set $r^{(n)}(0) = r(0) = 0.0561$, the value of r at time 0. Figure 4.4 shows $N = 5$ sample paths of $r^{(n)}(t)$.

Step3 : We have values $r^{(n)}(521), n = 1, \cdots, N$.

Step4 : Compute the bond values $p(r^{(n)}(521), 521), n = 1, \cdots, N$, and discount back to time that is 6 months remained before maturity ($t = 495$) using the generated values

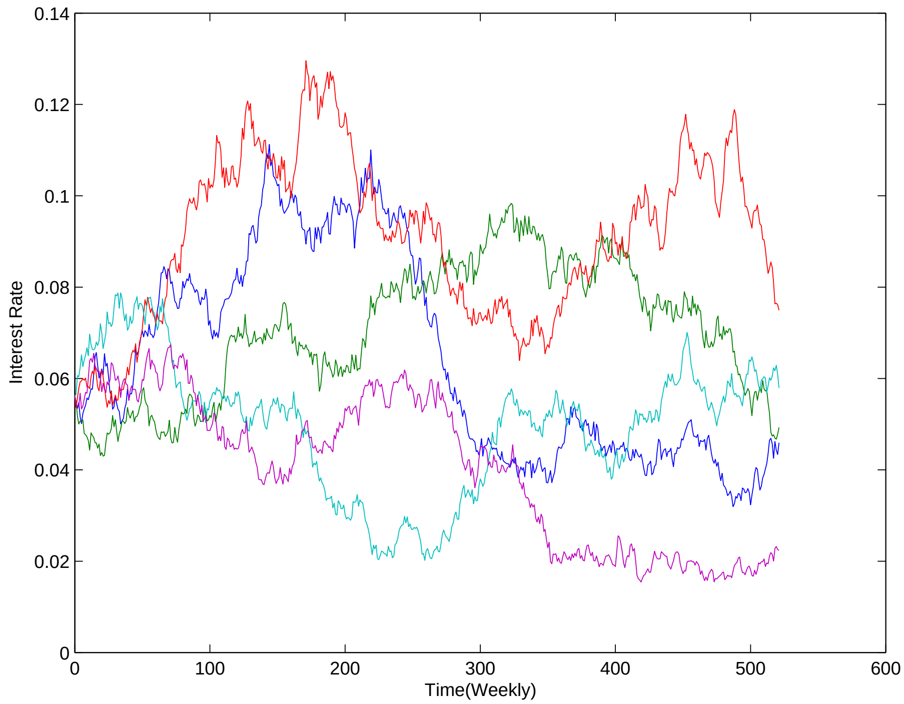

Figure 4.4: 5 sample paths of CIR interest rate process $r(t)$ with $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.

of the interest rate $r(t)$.

Step5 : The Monte Carlo estimate $\tilde{p}(r(495), 495)$ of the value $p(r(495), 495)$ at time 495 is

$$\tilde{p}(r(495), 495) = \frac{1}{N}\sum_{n=1}^{N} p^{(n)}(r(495), 495).$$

Step6 : Compute $V(r^{(n)}(495), 495), n = 1, \cdots, N$, and discount back to time $t = 235$ (5-Year call option) using the generated values of the interest rate $r(t)$; that is, compute

$$V^{(n)}(r^{(n)}(235), 235) = \exp\Big\{-\sum_{235}^{495} r^{(n)}(t)\Delta t\Big\}\Big(p\big(495, r^{(n)}(495)\big) - K\Big)^{+}, \quad n = 1, \cdots, N.$$

where K is exercise price.

Step7 : The Monte Carlo estimate $\widetilde{V}(r(235), 235)$ of the value $V(r(235), 235)$ at time 235 is

$$\widetilde{V}(r(235), 235) = \frac{1}{N}\sum_{n=1}^{N} V^{(n)}(r(235), 235).$$

Table 4.3 and Figure 4.5 shows the bond prices in different samples N. We can find that the bond prices change little after $N = 5000$ samples, but N larger than $N = 5000$ samples is profligate users of computing time and power. From now on, we take $N = 5000$. Table 4.4 shows the bond prices and the values of European call option in different exercise price are in Table 4.5.

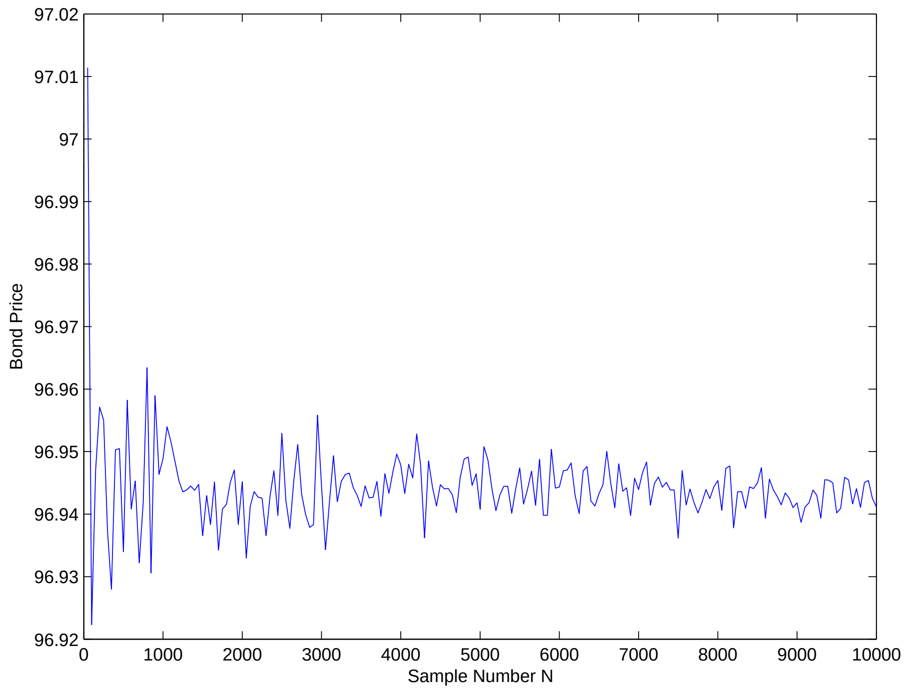

Figure 4.5: The bond prices in different samples N.

Current Interest Rate	Sample Number					
6%	1000	3000	5000	7000	9000	10000
Bond Price	96.9489	96.9445	96.9408	96.9439	96.9418	96.9411
CPU Time	0.2970	0.8750	1.4380	1.9850	2.5780	2.8440

Table 4.3: The bond prices and computing time in different samples N.

5000 Samples	Current Short-Term Interest Rate					
$100 face value	2%	4%	6%	8%	10%	12%
Bond Price	98.8741	97.9023	96.9452	95.9966	95.0509	94.1227
Analytic Solution	98.9127	97.9797	97.0555	96.1401	95.2332	94.3350

Table 4.4: The 10-Year bond prices that maturity time is after 6 months for the Cox, Ingersoll, and Ross model with $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.

5000 Samples	Current Short-Term Interest Rate					
Exercise Price	2%	4%	6%	8%	10%	12%
90	6.4478	5.8267	5.2438	4.6581	4.1804	3.7020
93	3.9528	3.4552	3.0051	2.5683	2.1998	1.8648
96	1.4441	1.0988	0.8389	0.6152	0.4341	0.2938

Table 4.5: Prices of a 5-Year call option on a 10-Year bond that maturity time is after 6 months for the the Cox, Ingersoll, and Ross model with $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.

Chapter 5

Conclusions

In this thesis, we introduce the manner of pricing interest rate derivatives. The explicit finite difference and Monte Carlo methods are main foundations. The numerical codes are implemented in Matlab. A numerical example of European call option on a U.S. 10-Y treasury bond are presented to illustrate. In this example, the computational results of the explicit finite difference method is better than Monte Carlo method.

We focused our work on particular one-factor interest rate models because of their great level of popularity among practitioners. However, when there is only one Brownian motion, the risk doesn't accurately estimate in the real market. This work can be extended to encompass multi-factor interest rates models. We believe that the multi-factor interest rates models are an interested topic for the future study.

References

[1] Phelim P. Boyle, Options : A Monte Carlo Approach, *Journal of Financial Economics, Vol. 4,* (1977) 323-338.

[2] Tomas Bjork, *Arbitrage Theory in Continuous Time,* Oxford, (1998).

[3] N.H. Bingham and R. Kiesel, *Risk-Neutral Valuation,* Springer, (2000).

[4] Martin Baxter, Andrew Rennie., *Financial calculus: an introduction to derivative pricing,* Cambridge University Press, (1996).

[5] John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross, A theory of the term structure of interest rates, *Econometrica, Vol. 53, No. 2,* (1985) 385-407.

[6] Dothan, L. U., On the Term Structure of Interest Rates, *Journal of Financial Economics, Vol. 6,* (1978) 59-69.

[7] Dothan, L. U., The Pricing of Options on Default-Free Bonds, *Journal of Financial and Quantitative Analysis, Vol. 17,* (1982) 75-100.

[8] G. Deelstra and F. Delbaen, Long-term returns in stochastic interest rate models: Convergence in law, *Stochastics Stochastics Rep., Vol. 55, No. 3-4,* (1995) 253-277 .

[9] M.A.H.Dempster and S.R.Pliska., *Mathematics of derivative securities,* Cambridge University Press, (1997).

[10] Griselda Deelstra and Gary Parker, *A Covariance Equivalent Discretisation of the CIR Model*, 5th Afir International Colloquium.

[11] Robert J. Elliott, P. Ekkehard Kopp., *Mathematics of financial markets,* Springer, (1999).

[12] Espen Gaarder Haug, *The complete guide to option pricing formulas,* McGraw-Hill, (1998).

[13] John C. Hull, *Options, Futures, and Other Derivatives*, Prentice Hall, (2002).

[14] John Hull, and Alan White, Valuing derivative securities using the explicit finite difference method, *Journal of Finance and Quantify Analysis, Vol 25, No. 1,* (1990) 87-100.

[15] Jessica James and Nick Webber., *Interest rate modelling,* John Wiley and Sons, (2000).

[16] Ralf Korn, and Elke Korn, *Option Pricing and Portfolio Optimization,* American Mathematical Society, (2000).

[17] Thomas Mikosch, *Elementary Stochastic Calculus,* Word Scientific, (1998).

[18] L.C.G. Rogers, Which model for term-structure of interest rates should one use, *Mathematical Finance, IMA Vol. 60,* Springer-Verlag, pp.93-112.

[19] Sidney I. Resnick, *A Probability Path,* BirkhAuser, (2001).

[20] Steven E. Shreve, *Stochastic Calculus and Finance*, (1997).

[21] Oldrich Vasicek, An equilibrium characterization of the term structure, *Journal of Finance, Vol 5,* (1977) 177-188.

[22] Paul Willmott, Sam Howison, Jeff Dewynne *The Mathematics of Financial Derivatives,* Cambridge University Press, (1995).

[23] Fang-Bo Yeh, Derivative Instruments: Mathematical Theory and Numerical Analysis, National Chiao Tung University Department of Applied Mathematics, Lecture Note of Introduction to Financial Mathematics.

[24] Fang-Bo Yeh, Financial Engineering: Theory and Applications, National Chiao Tung University Department of Applied Mathematics, Lecture Note of Introduction to Financial Mathematics (2000).

[25] Fang-Bo Yeh, National Chiao Tung University Department of Applied Mathematics, Lecture Note of Introduction to Financial Mathematics (2002).

[26] Fang-Bo Yeh, Tunghai University EMBA, Lecture Note of Financial Engineering (2003).

Appendix A

A.1 Estimation Method

The CIR model (1985) represented the dynamics of the short interest rate r by the stochastic differential equation:

$$dr(t) = k\Big(\mu - r(t)\Big)dt + \sigma\sqrt{r(t)}\ dW(t), \tag{A.1}$$

where $k > 0, \mu > 0, \sigma > 0$ are constants.
For $t < s$, the mean and variance of this distribution can be obtained as

$$\begin{aligned} E\big(r(s) \mid r(t)\big) &= \mu\big(1 - e^{-k(s-t)}\big) + \big(1 - e^{-k(s-t)}\big)r(t), \\ Var\big(r(s) \mid r(t)\big) &= \frac{\mu\sigma^2}{2k}\big(1 - e^{-2k(s-t)}\big)^2 + \frac{\sigma^2}{k}\big(e^{-k(s-t)} - e^{-2k(s-t)}\big)^2 r(t). \end{aligned} \tag{A.2}$$

Since the variance is a function of the state variable and is therefore time dependent, the more complicated variance structure leads to a more complicated estimation procedure such that the AR(1) process is no longer applicable. The regression model that produces the least square results can, however, be modified to produce reasonable estimates. For example, let us adopt the same regression model as the previous section. The model is expressed as

$$r_{t+\Delta t} = a + br_t + \varepsilon_t, \tag{A.3}$$

where $a = \mu(1 - e^{-k\Delta t})$ and $b = e^{-k\Delta t}$.

The error term ε is not identically and independently distributed. Rather, it is a function of the state variable; therefore, ordinary least squares (OLS) does not apply. With the variance structure specified in equation (A.2), however, equation (A.3) can be viewed as a regression model with heterology. By adopting the standard technique in Judge et al. (1982, 416), we can run weighted least squares as follows:

1. Run an OLS:
$$r_t = \beta_0 + \beta_1 r_{t-1} + \xi_t, \tag{A.4}$$
2. The variance of the error term must satisfy equation (A.2):
$$E\big(\xi_t^2\big) = \frac{\mu\sigma^2}{2k}\big(1 - e^{-2k\Delta t}\big)^2 + \frac{\sigma^2}{k}\big(e^{-k\Delta t} - e^{-2k\Delta t}\big)^2 r_{t-1}. \tag{A.5}$$
3. Run
$$E(\xi_t^2) = \alpha_0 + \alpha_1 r_{t-1} + \varpi_t, \tag{A.6}$$
where $\alpha_1 = \frac{\mu\sigma^2}{2k}(1 - e^{-2k\Delta t})^2$, and $\alpha_0 = \frac{\sigma^2}{k}(e^{-k\Delta t} - e^{-2k\Delta t})^2$.
4. Solve for k and μ from OLS and use α_0 or α_1 to solve for σ.

Note that both α_0 and α_1 can give σ. In this instance, we choose the one that generates an estimate more consistent with previous studies.

The price of risk is not identified because Treasury bill rates are used as a proxy for the instantaneous risk-free rate. In the fitting of the term structure, we can use the variable flexibly to adjust properly for theoretical prices.

A.2 Results

We will illustrate the method presented in the previous section by applying them to data consisting of (annualized) rates on American Government 10-year treasury note for 522 weeks covering the period of August 1983 until August 1993.

The results for the method of estimation of the parameters of the CIR model are presented in Table A.1. We chose the starting value $r_0 = 0.0561$, the last rate in our data set (August 1993). In Figure A.1, the real interest rate and the estimated interest rate show covering the period of August 1993 until August 2003.

10-Y Treasury Note						
β_0	β_1	α_0	α_1	μ	k	σ
0.0002317	0.9960626	-2.453E-06	5.948E-05	0.058856	0.205714	0.055855

Table A.1: Parameter estimates for the CIR model.

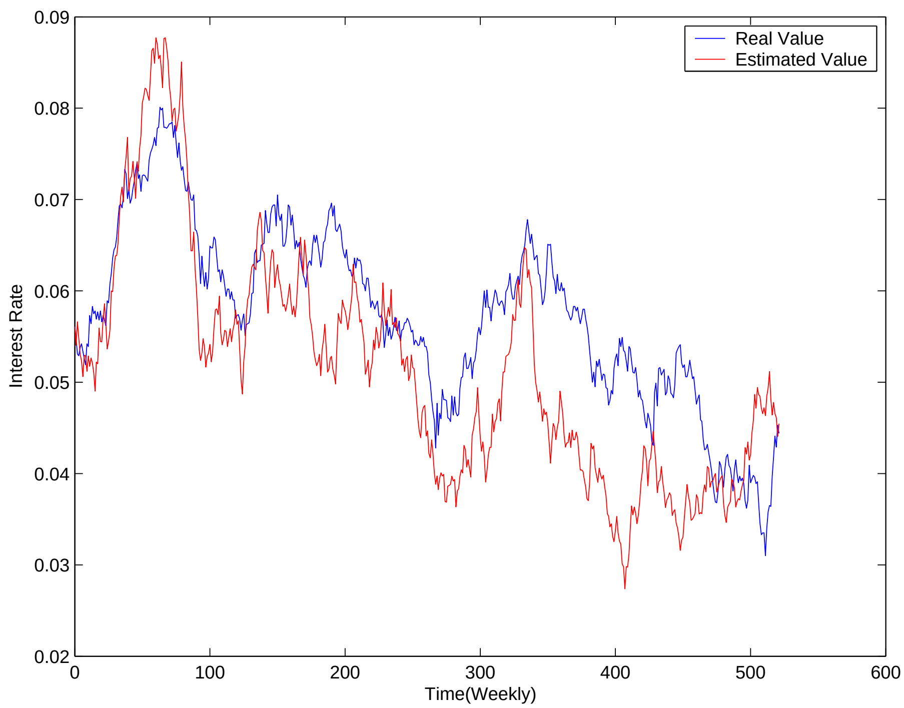

Figure A.1: The real interest rate v.s. the estimated interest rate with $k = 0.205714, \mu = 0.058856$, and $\sigma = 0.055855$.