Abstract

In 1998, *Rose, Suhov and Vvdenskaya* discussed a fully-connected packet-switched network with N nodes. Messages generated from a source i and deliver to a destination j are assumed as Poisson process $\Psi_{i,j}$ with rate v. Each message is divided into smaller units called packets. The lengths of messages are $i.i.d$ and bounded. The end-to-end delay time is the time that all packets be transmitted completely from its source to its destination. Each packet can choose a direct route $(i \to j)$ with probability p or an alternate route $(i \to k \to j)$ with probability q. They found the optimal policies about p and q with different means of message lengths and v such that the end-to-end delay time is minimized.

Here we use the same routing principle of this packet-switched network to simulate a transportation network. Consider a fully-connected traffic network with 2N+1 nodes, labelled with $1,\ 2,\ 3, ..., \ N, ..., \ 2N-2,\ \alpha,\ \beta,\ \gamma$. Suppose customers generated from a source node i and traversed to a destination node j is a Poisson process $\Psi_{i,j}$ with rate v for all different $i,\ j$ pairs. Each customer can choose to drive a car or to take a bus independently with probabilities p and q (=$1-p$) respectively. Customers who driving a car may drive on a direct route ($i \to j$) or on a specific alternate route ($i \to k \to j$) with probability m_1, and $\frac{m_2}{2N-1}$ respectively, where $m_2 = 1 - m_1$. Suppose for every K_1 minutes, there is a bus on a specific alternate route $\alpha \rightleftharpoons n_s \rightleftharpoons \beta$, or $\alpha \rightleftharpoons n_t \rightleftharpoons \gamma$, where $1 \le n_s \le N-1$, and $N \le n_t \le 2N-2$. The goal of this paper is to find optimal policies for probabilities m_1, m_2, p, and q such that the end-to-end delay time is minimized in this M/G/1 queuing network. We can also address the relationships between these probabilities. In reality, the capacity of a bus is finite. It is obvious that when the buses have finite capacities, the optimal probability of q (the probability that customers choose to take a bus) will decrease as the capacity of the bus decreases.

1 Introduction

There are ways to transmit data from a source to a destination in a network. *Rose, Suhov and Vvedenskaya et al.*(1998) [8] discussed a packet-switched network that uses the store-and-forward technique, whereby data is transmitted from its source to its destination without a complete circuit first being established between the source and the destination. In this packet-switched network, all messages are divided into smaller units called packets and transmitted in the network independently until they reach the destination node, where the original message is then reassembled. Each packet transmits independently in the network. Suppose that packets generated from a source often using different routes until they all reach the destination node. At the time when a message is generated, a route decision is made about which part of it will be routed on a direct path $(i \to j)$, and which part of it will be routed on an alternate path $(i \to k \to j)$. Define the end-to-end delay time to be the time which elapses as a message generated from a source until all its packets transmitted completely to the destination. Therefore, when all packets of the same message are transmitted at the same time, the end-to-end delay time (from its source to its destination) often tends to reduce.

Consider a fully-connected network with N nodes, messages with different source i and destination j are generated as an independent Poisson process $\Psi_{i,j}$ with rate v. Message lengths are independent identically distributed (i, i, d) and bounded. Each packet of a message can be transmitted on a direct route $(i \to j)$ or an alternate route $(i \to k \to j)$ from its source to its destination with probabilities p and $\frac{q}{N-2}$ respectively, where $q = 1-p$. All packets transmitted from the corresponding queues are operating on an FCFS basis. For this practical network model, a useful results could be obtained about the optimal policy which minimizes the end-to-end delay time. The results are listed in Table 1.1, where P_{opt} represents the optimal probability that a packet chooses a direct path, N is the number of nodes and v is the rate that messages are generated from a source node in the network.

$N < N'$	$v < v'$
$P_{opt}(N, v) \geq P_{opt}(N', v)$	$P_{opt}(N, v') \geq P_{opt}(N, v)$

Table 1.1

Based on the store-and-forward transmission technique of packet-switched network, we use this transmission method to simulate a transportation network. Every car or bus is driven on a path in the network as a packet. When customers generated from a source, they choose to drive a car or take a public transportation according to some probabilities. We need to choose a better probabilities such that the transportation network is more efficient, and customers spent less time traverse from its source to its destination. On the other hand, in heavy traffic conditions, if most of customers choose to take public transportation then it may reduce the end-to-end delay time.

Consider a fully-connected transportation network with $2N + 1$ nodes labelled with 1, 2, 3, ..., N, ..., $2N - 2$, α, β, and γ (see Figure 1.1), customers with different source and destination pairs i, j are generated as a Poisson process $\Psi_{i,j}$ with rate v, and these processes are $i.i.d.$ At the time when customers are generated, the number of customers L is bounded (also $i.i.d.$) and have probability distribution $b(l)$, $l = 1, 2, ..., m$. Customers act as messages from a source and the number of customers who choose to drive is as the number of packets of a message. Every customer may choose to drive or take a bus independently with probabilities p and q $(=1-p)$ respectively. Customers may choose to drive on a direct route $(i \rightarrow j)$ or a specific alternate route $(i \rightarrow k \rightarrow j)$ from the source i to the destination j with probabilities m_1 and $\frac{m_2}{2N-1}$ respectively, where $m_2 = 1 - m_1$. Once customers are generated from a source, the routing policy is also made on how many customers should drive (on a direct path or a specific alternate path) and how many customers should take a bus. Suppose for every K_1 minutes, there will be a bus start from α, β, and γ on a specific alternate route $\alpha \rightleftharpoons n_s \rightleftharpoons \beta$ or $\alpha \rightleftharpoons n_t \rightleftharpoons \gamma$, where $1 \leq n_s \leq N-1$, and $N \leq n_t \leq 2N-2$. Let $W_i(d)$ be the probability that the waiting time for a bus is d at node i. It is reasonable to assume that the probability of the waiting time for a bus is 0 at node α is 1 when the number of nodes in the network is infinite. Also the transmission time between any two nodes is assumed as a constant time T.

Figure 1.1: Fully-Connected Transportation Network with 2N+1 nodes

Define the end-to-end delay time D^0 to be the time as customers generated from a source until they all reached the destination (no matter they drive or take a bus). For example if three customers A, B, and C move from a node i to a node j. A, B spent two and three hours respectively to drive, C spent four hours to take a bus. Then the end-to-end delay is four hours. Suppose l customers generated from a source and the sth $(1 \leq s \leq l)$ customer takes T_s time to move from the source to the destination, the end-to-end delay time $D^0 = \max_s T_s$. The distribution function corresponding to the end-to-end delay time is denoted by

$$F(x, N) = P(D^0 \leq x). \tag{1.1}$$

According to the definition, we can derive the limiting distribution function

$$F(x) = \lim_{N \to \infty} F(x, N). \tag{1.2}$$

Since the driver can choose a direct route (called 1-path), or an alternate route (called 2-path) from a source to a destination, the driver may be waiting on the 1-path, or waiting on the first link of the 2-path (called 2-1 path), or waiting on the second link of the 2-path (called 2-2 path). We denote these waiting times by W, W', and W'' respectively. Then W, W', and W'' are independent random variables in a stationary M/G/1 queue (see Figure 1.2) [9, 10]. All cars joining the queues are on an FCFS basis.

Figure 1.2: Waiting time distributions on a 1-path and 2-path

Let $F_{\alpha,\beta}(x)$ denotes the limiting distribution function that customers traverse from node α to node β and let $F_{u,\beta}(x)$ denotes the limiting distribution function that customers traverse from node u to node β. At node α, customers will have waiting time W and transmission time iT if i customers choose to drive on the 1-path from the source node α to the destination node β. Since a customer chooses a 2-path from a source to a destination with probability m_2, two customers route on the same 2-path have probability $\frac{(m_2)^2}{2N-1}$. As $N \to \infty$, the probability that two customers joining the same 2-path will approach 0. So the transmission time is $2T$ on every 2-path. If customers choose to take a bus from a source node α to a destination node β, they will have waiting times $W^{'}$, and $W^{''}$ on the 2-1 and 2-2 path and the waiting time W_α for waiting a bus at node α. We can derive the limiting distribution function $F_{\alpha,\beta}(x)$ first as

$$\begin{aligned} F_{\alpha,\beta}(x) =& \sum_{i=1}^{m} b(i) p^i [m_1^i P(W+iT \le x) + \sum_{j=1}^{i} \binom{i}{j} m_1^{i-j} m_2^j P(W+(i-j)T \le x) P(W^{'}+W^{''}+2T \le x)^j] \\ &+ \sum_{k=1}^{m-1} \sum_{i=k+1}^{m} b(i) \binom{i}{k} p^{i-k} q^k [m_1^{i-k} P(W+(i-k)T \le x) + \sum_{j=1}^{i-k} \binom{i-k}{j} m_1^{i-k-j} m_2^j P(W+(i-k-j)T \le x) \\ &\quad P(W^{'}+W^{''}+2T \le x)^j] P(W^{'}+W^{''}+W_\alpha+2T \le x) \\ &+ \sum_{i=1}^{m} b(i) P(W^{'}+W^{''}+W_\alpha+2T \le x). \end{aligned} \tag{1.3}$$

At node u $(1 \le u \le N-1)$, customers will have waiting time W and transmission time iT if i customers choose to drive on a 1-path from the source node u to the destination node β or customers will have waiting time $W^{'}$, and $W^{''}$ corresponding to the first link of a 2-path and the second link of a 2-path and transmission time $2T$ if customers choose to drive on a 2-path from the source node u to the destination node β. On the other hand, if customers choose to take a bus from a source node u to a destination node β, they will have waiting time $W^{''}$ on the 2-2 path of a bus and the waiting time W_u for waiting a bus at node u. From these we can also find the limiting distribution function $F_{u,\beta}(x)$ as

$$F_{u,\beta}(x)= \sum_{i=1}^{m} b(i)p^i[m_1^i P(W+iT\leq x)+\sum_{j=1}^{i} \binom{i}{j} m_1^{i-j} m_2^j P(W+(i\text{-}j)T\leq x)P(W^{'}+W^{''}+2T\leq x)^j]$$
$$+\sum_{k=1}^{m-1} \sum_{i=k+1}^{m} b(i)\binom{i}{k} p^{i-k} q^k [m_1^{i-k} P(W+(i\text{-}k)T\leq x)+$$
$$\sum_{j=1}^{i-k} \binom{i-k}{j} m_1^{i-k-j} m_2^j P(W+(i\text{-}k\text{-}j)T\leq x)P(W^{'}+W^{''}+2T\leq x)^j]P(W^{''}+W_u+T\leq x)$$
$$+\sum_{i=1}^{m} b(i)q^i P(W^{''}+W_u+T\leq x). \tag{1.4}$$

According to equations (1.3) and (1.4) we can then find the probabilities p, q, $m_{1,}$ and m_2 to minimize the end-to-end delay time D^0.

In equations (1.3) and (1.4), we have assumed that the network has infinite node and all buses have infinite capacity. In section 4 and section 5, we will discuss the cases that the network contains only finite node (with unlimited bus capacity) or the bus has finite capacity (with unlimited node in the network) respectively. It is obvious that the optimal probabilities of q (the probability that a customer chooses to take a bus) and m_2 decrease as network nodes and capacity of a bus decrease.

2 Theoretical Analyses

To complete equations (1.3) and (1.4), we will define some traffic parameters. In this transportation network, each bus has its route and every car can choose 1-path or 2-path independently. To analyze car flows on each link, we will discuss case (I) that the network has finite node, and case (II) that the network has unlimited node.

(I) There are finite node in the network

There may have more than one car depart from a source and choose the same path, either 1-path or 2-path. Suppose that k customers generated from a source node i choose the same link $\overrightarrow{ih}$. Let $\lambda_{k,N}(1)$ be the 1-path car flows with k customers joining this link $\overrightarrow{ih}$, and let $\lambda_{k,N}(2)$ be the 2-path car flows with k customers joining this link $\overrightarrow{ih}$. Then the total arrival rate from node i to node h which has no bus passing is

$$\begin{aligned}\lambda_N^{Sum} &= \lambda_N^{Sum}(1) + \lambda_N^{Sum}(2) \text{ , with} \\ \lambda_N^{Sum}(1) &= \textstyle\sum_{k=1}^{m} \lambda_{k,N}(1) \\ &= \textstyle\sum_{k=1}^{m} \sum_{i=k}^{m} b(i)\binom{i}{k}(pm_1)^k(1-pm_1)^{i-k}v, \text{ and} \\ \lambda_N^{Sum}(2) &= \textstyle\sum_{k=1}^{m} \lambda_{k,N}(2) \\ &= \textstyle\sum_{k=1}^{m} \sum_{i=k}^{m} 2b(i)\binom{i}{k}p^k m_2(m_{2,N})^{k-1}(1-pm_{2,N})^{i-k}v,\end{aligned} \tag{2.1}$$

where $m_{2,N} = \frac{m_2}{2N-1}$, $\lambda_N^{Sum}(1)$ is the total arrival rate of 1-path car flows and $\lambda_N^{Sum}(2)$ is the total arrival rate of 2-path car flows.

The total arrival rate on $\overrightarrow{ih}$ which has a bus passing is

$$\lambda_N^{B,Sum} = \lambda_N^{Sum} + \frac{1}{K_1}, \text{ where } \frac{1}{K_1} \text{ is the arrival rate of a bus.} \tag{2.2}$$

To analyze 2-path car flows $\lambda_N^{Sum}(2)$, let $\lambda_{k,N}(2_1)$ be the 2-1 path car flows with k customers on the link $\overrightarrow{ih}$. The model considered is a fully-connected graph with $2N+1$ nodes, so the total number of 2-paths with $\overrightarrow{ih}$ as its first link is $2N-1$. It is possible that more than one car been driven on this 2-1 path, we are able to list all of the possible arrival rates that customers join this 2-1 path.

When $k = 1$, the arrival rate of car flows on a 2-1 path from node i to node h is

$$\lambda_{1,N}(2_1) = (2N-1)\sum_{i=1}^{m} b(i)\binom{i}{1}pm_{2,N}(1-pm_{2,N})^{i-1}v. \qquad (2.3)$$

When $k = 2$, the arrival rate of car flows on a 2-1 path from node i to node h is

$$\lambda_{2,N}(2_1) = (2N-1)\sum_{i=1}^{m} b(i)\binom{i}{2}p^2(m_{2,N})^2(1-pm_{2,N})^{i-2}v. \qquad (2.4)$$

In general, for any $k = 1, 2, ..., m$, the arrival rate of car flows on a 2-1 path from node i to node h is

$$\begin{aligned}\lambda_{k,N}(2_1) &= (2N-1)\sum_{i=1}^{m} b(i)\binom{i}{k}p^k(m_{2,N})^k(1-pm_{2,N})^{i-k}v.\\ &= (2N-1)\sum_{i=k}^{m} b(i)\binom{i}{k}\, p^k(m_{2,N})^k(1-pm_{2,N})^{i-k}v.\\ &= \sum_{i=k}^{m} b(i)\binom{i}{k}p^k m_2(m_{2,N})^{k-1}(1-pm_{2,N})^{i-k}v. \qquad (2.5)\end{aligned}$$

So the total arrival rate of 2-1 path car flows at the node h is

$$\lambda_N^{Sum}(2_1) = \sum_{k=1}^{m}\sum_{i=k}^{m} b(i)\binom{i}{k}\, p^k m_2(m_{2,N})^{k-1}(1-pm_{2,N})^{i-k}v. \qquad (2.6)$$

Similarly, the total arrival rate of 2-2 path car flows at the node h is

$$\lambda_N^{Sum}(2_2) = \sum_{k=1}^{m}\sum_{i=k}^{m} b(i)\binom{i}{k}\, p^k m_2(m_{2,N})^{k-1}(1-pm_{2,N})^{i-k}v. \qquad (2.7)$$

From equations (2.6) and (2.7), the total arrival rate of 2-path car flows as in equations (2.1) and (2.2) at the node h is then

$$\begin{aligned}\lambda_N^{Sum}(2) &= \lambda_N^{Sum}(2_1) + \lambda_N^{Sum}(2_2)\\ &= \sum_{k=1}^{m}\sum_{i=k}^{m} 2b(i)\binom{i}{k}\, p^k m_2(m_{2,N})^{k-1}(1-pm_{2,N})^{i-k}v. \qquad (2.8)\end{aligned}$$

(II) There are unlimited node in the network

Let λ_∞^{Sum}, $\lambda_\infty^{Sum}(1)$, $\lambda_\infty^{Sum}(2)$ be the total arrival rate, total 1-path car flows and total 2-path car flows respectively on link $\overrightarrow{ih}$ which has no bus passing. Then $\lambda_\infty^{Sum} = \lambda_\infty^{Sum}(1) + \lambda_\infty^{Sum}(2)$.

From equation (2.1), the total arrival rate λ_∞^{Sum} from node i to node h which has no bus passing is

$$\begin{aligned}\lambda_\infty^{Sum} &= \lim_{N\to\infty}\lambda_N^{Sum}\\ &= \lim_{N\to\infty}[\lambda_N^{Sum}(1) + \lambda_N^{Sum}(2)]\\ &= \lim_{N\to\infty}\sum_{k=1}^{m}\lambda_{k,N}(1) + \lim_{N\to\infty}\sum_{k=1}^{m}\lambda_{k,N}(2)\end{aligned}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{m} \sum_{i=k}^{m} b(i)\binom{i}{k}(pm_1)^k(1-pm_1)^{i-k}v+$$

$$\lim_{N\to\infty} \sum_{k=1}^{m} \sum_{i=k}^{m} 2b(i)\binom{i}{k}p^k m_2(m_{2,N})^{k-1}(1-pm_{2,N})^{i-k}v.$$

$$= \sum_{k=1}^{m} \sum_{i=k}^{m} b(i)\binom{i}{k}(pm_1)^k(1-pm_1)^{i-k}v + \sum_{i=1}^{m} 2b(i)\binom{i}{1}\, pm_2\, v. \tag{2.9}$$

Let $\lambda_{k,\infty}(1)$, and $\lambda_{k,\infty}(2)$ be the 1-path and 2-path car flows with k customers. Then $\lambda_{k,\infty}(1) = \lim_{N\to\infty} \lambda_{k,N}(1)$, and $\lambda_{k,\infty}(2) = \lim_{N\to\infty} \lambda_{k,N}(2)$. From equations (2.1) and (2.8), we obtain that

$$\lambda_{k,\infty}(1) = \lim_{N\to\infty} \lambda_{k,\,N}(1)$$
$$= \sum_{i=k}^{m} b(i)\binom{i}{k}p^k m_1^k(1-pm_1)^{i-k}v \quad \text{if } k = 1,\ 2,\ 3,\ \ldots\ m, \text{ and} \tag{2.10}$$

$$\lambda_{k,\infty}(2) = \lim_{N\to\infty} \lambda_{k,\,N}(2)$$
$$= \begin{cases} 2\sum_{i=1}^{m} b(i)\binom{i}{1}\, pm_2\, v & \text{if } k = 1, \\ 0 & \text{if } k = 2,\ 3,\ \ldots,\ m. \end{cases} \tag{2.11}$$

The total arrival rate $\lambda_{\infty}^{B,Sum}$ from node i to node h which has a bus passing is

$$\lambda_{\infty}^{Sum} + \frac{1}{K_1}. \tag{2.12}$$

From equation (2.11), the total arrival rate $\lambda_{\infty}^{Sum}(2)$ of 2-path (2-1 path and 2-2 path) from node i to node h is

$$\sum_{i=1}^{m} 2\ b(i)\binom{i}{1}pm_2v = 2pm_2v\sum_{i=1}^{m} b(i)\binom{i}{1}$$
$$= 2pm_2v\sum_{i=1}^{m} b(i)i$$
$$= 2pm_2vE(L). \tag{2.13}$$

From equations (2.10), (2.11) and (2.13), the total arrival rate λ_{∞}^{Sum} on a link that has no bus passing is

$$\lambda_{\infty}^{Sum} = \sum_{k=1}^{m}[\lambda_{k,\infty}(1) + \lambda_{k,\infty}(2)]$$

$$= \sum_{k=1}^{m} \sum_{i=k}^{m} b(i)\binom{i}{k}p^k m_1^k(1-pm_1)^{i-k}v + 2pm_2vE(L)$$

$$= \sum_{i=1}^{m} b(i)\binom{i}{1}pm_1(1-pm_1)^{i-1}v + \sum_{i=2}^{m} b(i)\binom{i}{2}(pm_1)^2(1-pm_1)^{i-2}v + \cdots$$

$$+\sum_{i=m-1}^{m} b(i)\binom{i}{m-1}(pm_1)^{m-1}(1-pm_1)^{i-m+1}v+\sum_{i=m}^{m} b(i)\binom{i}{m}(pm_1)^{m}(1-pm_1)^{i-m}v$$

$$+2pm_2vE(L)$$

$$= [\ b(1)(pm_1)^1\ +b(2)\binom{2}{1}(pm_1)^1(1-pm_1)+b(3)\binom{3}{1}(pm_1)^1(1-pm_1)^2+$$

$$\cdots+b(m)\binom{m}{1}(pm_1)^1(1-pm_1)^{m-1}]v$$

$$+[b(2)(pm_1)^2+b(3)\binom{3}{2}(pm_1)^2(1-pm_1)+b(4)\binom{4}{2}(pm_1)^2(1-pm_1)^2+$$

$$\cdots+b(m)\binom{m}{2}(pm_1)^2(1-pm_1)^{m-2}]v$$

$$+[b(3)(pm_1)^3+b(4)\binom{4}{3}(pm_1)^3(1-pm_1)+b(5)\binom{5}{3}(pm_1)^3(1-pm_1)^2+$$

$$\cdots+b(m)\binom{m}{3}(pm_1)^3(1-pm_1)^{m-3}]v$$

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$$+b(m)(pm_1)^m v$$

$$+2pm_2vE(L)$$

$$= b(1)pm_1v$$

$$+b(2)(\binom{2}{1}pm_1(1-pm_1)+(pm_1)^2)v$$

$$+b(3)(\binom{3}{1}pm_1(1-pm_1)^2+\binom{3}{2}(pm_1)^2(1-pm_1)+(pm_1)^3)v$$

$$+b(4)(\binom{4}{1}pm_1(1-pm_1)^3+\binom{4}{2}(pm_1)^2(1-pm_1)^2+\binom{4}{3}(pm_1)^3(1-pm_1)+(pm_1)^4)v$$

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$$+b(m)(\binom{m}{1}pm_1(1-pm_1)^{m-1}+\binom{m}{2}(pm_1)^2(1-pm_1)^{m-2}+\binom{m}{3}(pm_1)^3(1-pm_1)^{m-3}+$$

$$\cdots+(pm_1)^m)v+2pm_2vE(L)$$

$$= b(1)v-b(1)(1-pm_1)v+b(2)[1-(1-pm_1)^2]v+b(3)[1-(1-pm_1)^3]v+b(4)[1-$$

$(1-pm_1)^4]v + \cdots + b(m)[1-(1-pm_1)^m]v + 2pm_2vE(L)$

$= [b(1)+b(2)+b(3)+b(4)+\cdots+b(m)]v - [b(1)(1-pm_1)+b(2)(1-pm_1)^2+b(3)(1-pm_1)^3+$

$\cdots + b(m)(1-pm_1)^m]v + 2pm_2vE(L)$

$$= (1-E[(1-pm_1)^L])v + 2pm_2vE(L). \tag{2.14}$$

Similarly, the total arrival rate $\lambda_\infty^{B,Sum}$ on a link that has a bus passing is

$$\lambda_\infty^{B,Sum} = \lambda_\infty^{Sum} + \tfrac{1}{K_1}$$
$$= (1-E[(1-pm_1)^L])v + 2pm_2vE(L) + \tfrac{1}{K_1}. \tag{2.15}$$

In this queuing system, the transmission time between any two adjacent nodes i and j (says link $\overrightarrow{ij}$) is always T. This means each car has service time T on any link $\overrightarrow{ij}$. Therefore, the service rate on any link $\overrightarrow{ij}$ is $1/T$. If the service rate is less than the arrival rate on a link, then a car may wait on this link forever, and the limiting distribution functions (1.3) and (1.4) would not converge. Therefore, for this queuing model to work well, the service rate must be larger than the arrival rate on each link, thus we have the following theorem.

Theorem 2.1 Let $\rho_1 = T(2-m_1)pvE(L)$ and $\rho_2 = T[(2-m_1)pvE(L) + 1/K_1]$, then $\rho_1 < 1$ and $\rho_2 < 1$. (ρ_1 is the traffic intensity on a link that has no bus passing and ρ_2 is the traffic intensity on a link that has a bus passing.)

Proof: We start by analyzing the maximal arrival rate on a link $\overrightarrow{ij}$ (which has no bus passing). From equations (2.9) and (2.13) the total arrival rate on a link $\overrightarrow{ij}$ is

$$\lambda_\infty^{Sum} = \textstyle\sum_{k=1}^m \sum_{i=k}^m b(i)\binom{i}{k}(pm_1)^k(1-pm_1)^{i-k}v + 2pm_2vE(L)$$
$$= \textstyle\sum_{i=1}^m b(i)\binom{i}{1}(pm_1)^1(1-pm_1)^{i-1}v$$
$$+ \textstyle\sum_{i=2}^m b(i)\binom{i}{2}(pm_1)^2(1-pm_1)^{i-2}v$$
$$\cdot$$
$$\cdot$$
$$+ \textstyle\sum_{i=m}^m b(i)\binom{i}{m}(pm_1)^m(1-pm_1)^{i-m}v$$
$$+2pm_2vE(L)$$
$$= b(1)pm_1v + b(2)\binom{2}{1}pm_1(1-pm_1)v + b(3)\binom{3}{1}pm_1(1-pm_1)^2v + \cdots$$

$+b(m)\binom{m}{1}pm_1(1-pm_1)^{m-1}v$

$+b(2)\binom{2}{2}(pm_1)^2v+b(3)\binom{3}{2}(pm_1)^2(1-pm_1)v+\cdots+b(m)\binom{m}{2}(pm_1)^2(1-pm_1)^{m-2}v$

$+b(3)\binom{3}{3}(pm_1)^3v\ +b(4)\binom{4}{3}(pm_1)^3(1-pm_1)v+\ \cdots+b(m)\binom{m}{3}(pm_1)^3(1-pm_1)^{m-3}v$

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$+b(m)(pm_1)^mv$

$+2pm_2vE(L)$

$=b(1)pm_1v+b(2)pm_1(\,2-2pm_1+pm_1)v\ +b(3)pm_1(\,3-6pm_1+3(pm_1)^2+3pm_1-$

$3(pm_1)^2+(pm_1)^2)v\ +\ \cdots\ +\ b(m)pm_1[\binom{m}{1}(1-pm_1)^{m-1}\ +\cdots+\ (pm_1)^{m-1}]+$
$2\,pm_2vE(L)$

$=b(1)pm_1v+b(2)pm_1(2-pm_1)v+b(3)pm_1(3-6pm_1+(pm_1)^2)v\ +\cdots+b(m)pm_1(m-$

$m(m-1)pm_1+\cdots-\cdots+(pm_1)^{m-1})v+2pm_2vE(L)$

$\leq b(1)pm_1v+2b(2)pm_1v+3b(3)pm_1v+\ \cdots+mb(m)pm_1v+2pm_2vE(L)$

$=pm_1v(b(1)+2b(2)+3b(3)+\cdots+mb(m))+\ 2pm_2vE(L)$

$=pm_1vE(L)+2pm_2vE(L)$

$=(m_1+2m_2)pvE(L)$

$$=(2-m_1)pvE(L). \qquad (2.16)$$

From equations (2.14) and (2.15), $\lambda_\infty^{Sum}=(1-E[(1-pm_1)^L])v+2pm_2vE(L)$ (the total arrival rate on a link that has no bus passing), and $\lambda_\infty^{B,Sum}=(1-E[(1-pm_1)^L])v+2pm_2vE(L)+\frac{1}{K_1}$ (the total arrival rate on a link that has a bus passing). Since the maximal arrival rate of λ_∞^{Sum} is $(2-m_1)pvE(L)$, so the maximal arrival rate of $\lambda_\infty^{B,Sum}$ is

$$(2-m_1)pvE(L)+\tfrac{1}{K_1}. \qquad (2.17)$$

From equation (2.16), the maximal arrival rate on a link which has no bus passing is $(2-m_1)pvE(L)$, and the service rate on each link is $1/T$. To avoid a link be always busy, $1/T > (2-m_1)pvE(L)$ is needed. And to avoid the link that has a bus passing been always busy, $1/T > (2-m_1)pvE(L) + \frac{1}{K_1}$ is needed from equation (2.17). Then the traffic intensities $\rho_1 = (2-m_1)pvE(L) \div 1/T = T(2-m_1)pvE(L) < 1$, and $\rho_2 = [(2-m_1)pvE(L)+1/K_1] \div 1/T = T[(2-m_1)pvE(L)+1/K_1] < 1$ are obtained corresponding to the link that has no bus passing and the link that has a bus passing. □

Since buses are scheduled every K_1 minutes, it is clear that the waiting time distribution, $B(x)$, for a bus is uniformly distributed and can be represented as $\int_0^{K_1} \frac{1}{x} dx = 1$.

Rose *et al.* [10] have considered the problem of finding the waiting time distribution, $G(x)$, for the stationary M/G/1 queue. The Kolmogorov equation [5]

$$G'(x) = \sum_{k=1}^{m} \lambda_k [G(x) - G(x-k)], \tag{2.18}$$

and its solution

$$G(x) = (1-\rho) \sum_{r_j \geq 0 : \sum_j j r_j \leq \lfloor x \rfloor} \prod_{j:\lambda_j > 0} e^{\lambda_j (x - \sum_j j r_j)} \frac{[-\lambda_j (x - \sum_j j r_j)]^{r_j}}{r_j!} \tag{2.19}$$

were found.

For limiting distribution functions (1.3) and (1.4) to be exist, we must have $P(W \leq x)$, $P(W' + W'' \leq x)$ and $P(W' + W'' + W_{\alpha(u)} \leq x)$. To compute the value of $G(x)$ we need to determine the indices $\{r_j\}$ in equation (2.19). A recursive procedure can be used to find $\{r_j\}$ which satisfies the inequality $\sum_j j r_j \leq \lfloor x \rfloor$. For example, to determine $\{r_j\}$ that satisfies $\sum_1^1 j r_j \leq \lfloor 3 \rfloor$ (*i.e.* $r_1 \leq 3$), it is obvious that $r_1 = \{0, 1, 2, 3\}$. To determine $\{r_j\}$ that satisfies $\sum_1^2 j r_j \leq \lfloor 3 \rfloor$. It is easy to solve the inequality and list its solution in the table

r_1	0	0	1	1	2	3
r_2	0	1	0	1	0	0
$r_1 + 2r_2$	0	2	1	3	2	3

$(r_1, r_2) = (0,0), (0,1), (1,0), (1,1), (2,0), (3,0)$

Table 2.1

To determine $\{r_j\}$ that satisfies $\sum_1^3 jr_j \leq \lfloor 3 \rfloor$, set $r_1 + 2r_2 = a$ and solve $a + 3r_3 \leq \lfloor 3 \rfloor$ using Table 2.1. The solutions are listed in Table 2.2, and Table 2.3.

a	0	0	1	2	3
r_3	0	1	0	0	0
$a + 3r_3$	0	3	1	2	3

Table 2.2

(r_1, r_2)	$(0,0)$	$(0,0)$	$(1,0)$	$(0,1),(2,0)$	$(1,1),(3,0)$
r_3	0	1	0	0	0
(r_1, r_2, r_3)	$(0,0,0)$	$(0,0,1)$	$(1,0,0)$	$(0,1,0),(2,0,0)$	$(1,1,0),(3,0,0)$

Table 2.3

$G(x)$ can then be computed by the above procedures. We can rewrite (1.3), (1.4) in terms of $G(x)$. Since $P(W+iT \leq x) = P(W \leq x-iT) = G(x-iT)$, $P(W'+W''+2T \leq x) = P(W'+W'' \leq x-2T) = G*G'(x-2T)$, $P(W'+W''+W_\alpha+2T \leq x) = P(W'+W''+W_\alpha \leq x-2T) = G*G'*B(x-2T)$.

Therefore, the limiting distribution functions (1.3) and (1.4) can be rewritten as

$$\begin{aligned}
F_{\alpha,\beta}(\text{x}) = & \sum_{i=1}^{m} b(i)\ p^i[m_1^i G(\text{x-}iT) + \sum_{j=1}^{i} \tbinom{i}{j} m_1^{i-j} m_2^j G(\text{x-(i-j)}T) G*G'(\text{x-2}T)^j] \\
& + \sum_{k=1}^{m-1} \sum_{i=k+1}^{m} b(i) \tbinom{i}{k} p^{i-k} q^k [m_1^{i-k} G(\text{x-(i-k)}T) + \\
& \sum_{j=1}^{i-k} \tbinom{i-k}{j} m_1^{i-k-j} m_2^j G(\text{x-(i-k-j)}T) G*G'(\text{x-2}T)^j] G*G'*B(\text{x-2}T) \\
& + \sum_{i=1}^{m} b(i) G*G'*B(x-2T), \text{ and}
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
F_{u,\beta}(\text{x}) = & \sum_{i=1}^{m} b(i) p^i[m_1^i G(\text{x-}iT) + \sum_{j=1}^{i} \tbinom{i}{j} m_1^{i-j} m_2^j G(\text{x-(i-j)}T) G*G'(\text{x-2}T)^j] \\
& + \sum_{k=1}^{m-1} \sum_{i=k+1}^{m} b(i) \tbinom{i}{k} p^{i-k} q^k [m_1^{i-k} G(\text{x-(i-k)}T) + \\
& \sum_{j=1}^{i-k} \tbinom{i-k}{j} m_1^{i-k-j} m_2^j G(\text{x-(i-k-j)}T) G{*}G'(\text{x-2}T)^j] G'*B(\text{x-2}T) \\
& + \sum_{i=1}^{m} b(i) G'*B(x-2T)
\end{aligned} \tag{2.21}$$

By definition, the convolution integrals $G * G(x) = \int_0^x G(x-u)dG(u)$, and $G * G * B(x) = \int_0^x G * G(x-u)dB(u)$ [8]. Hence $G(x)$, $G * G(x)$ and $G * G * B(x)$ are computed, the limiting distribution functions (1.3) and (1.4) can then be obtained.

3 Optimal Policies for a Network with Unlimited Node and Infinite Capacity

Suppose that the capacity of a bus is infinite and the network has infinite node. Assume that the waiting time for a bus is 0 at node α when the number of nodes in the network is infinite. We will discuss the limiting distribution function for different values p, q, m_1, m_2 and K_1. Assume that $v = 0.05$, $T = 1$, and the number of customers generated are uniformly distributed on $\{1, 2, 3, ..., 10\}$. Figure $3.1 - 3.17$ provide some simulation results of $F_{\alpha,\beta}(x)$ and $F_{u,\beta}(x)$. We will discuss cases for $K_1 = 18$, $K_1 = 10$ and $K_1 = 5$.

(I) $K_1 = 18$

At node α, the arrival rate of buses is small, so the traffic intensity on links of 2-path are just a little larger than that on the 1-path. We observe that:

(i) The traffic intensity on each link of 2-path is just a little larger than that on the 1-path. It may have advantages to choose a 2-path if many customers generated.

(ii) When $m_1 = 0.2$, $F_{\alpha,\beta}(x)$ for $p = 0.2 > F_{\alpha,\beta}(x)$ for $p = 0.4$

$> F_{\alpha,\beta}(x)$ for $p = 0.6$

$> F_{\alpha,\beta}(x)$ for $p = 0.8$

$> F_{\alpha,\beta}(x)$ for $p = 1$.

This says as the probability of p increases, the limiting distribution function $F_{\alpha,\beta}(x)$ converges more slowly (see Figure 3.1).

(iii) As the probability of q increases, the car flows on each 2-path will be reduced, therefore, customers who choose to drive will have more advantages to drive on $2-$ paths.

(iv) When customers all take a bus, the waiting time for a bus is 0, therefore, to take a bus is just like to drive a car on themselves to the destination. Therefore, the optimal policy is $p = 0$.

At node u, the traffic intensity on links of 2-path is smaller than the traffic intensity on the 1-path (from node u to node β). We observe that:

(i) It is better to drive than to take a bus, because the waiting time for a bus is long. Simulation results shows that

$$
\begin{aligned}
F_{u,\beta}(x) \text{ for } p = 1 > F_{u,\beta}(x) &\text{ for } p = 0.6 \\
> F_{u,\beta}(x) &\text{ for } p = 0.4 \\
> F_{u,\beta}(x) &\text{ for } p = 0.2
\end{aligned}
$$

In other word, $F_{u,\beta}(x)$ decreases as p decreases (see Figure 3.2).

(ii) It will have advantages to drive on a 2-path. As an example, Figure 3.3 shows that $F_{u,\beta}(x)$ for $p = 0.5 \ < \ F_{u,\beta}(x)$ for $p = 0.2$ if $x > 5$. Therefore, customers should drive on a 2-path if the number of customers generated from a source is not small.

From (i) and (ii), customers should have advantages to drive on 2-paths.

Figure 3.1: $K_1 = 18$, node α, $m_1 = 0.2$, $p = 0.2, 0.4, 0.6, 0.8, 1$

Figure 3.2: $K_1 = 18$, node u, $m_1 = 0.2$, $p = 0.6, 0.4, 0.2, 1$

Figure 3.3: $K_1 = 18$, node u, $p = 0.6$, $m_1 = 0.2, 0.5$

(II) $K_1 = 10$

For $K_1 = 10$, the total arrival rate on a link which has a bus passing is larger than that for $K_1 = 18$. At node α, if customers choose to drive cars on a 2-path, they may drive either on links of 2-path which has a bus passing or on its second link of 2-path which has a bus passing. Hence on links of 2-path from node α to node β there still have larger arrival rate than the 1-path. We observe that:

(i) When all customers drive cars, the traffic intensity on links of 2-paths is large. T-herefore, if there are not many customers generated from a source, customers shou-ld choose a 1-path.

(ii) As the probability of p decreases, more customers choose to take a bus and traffic intensity will reduce. Therefore, it has advantages to drive on 2-paths as p decreases (see Figure $3.4 - 3.5$).

(iii) When $m_1 = 0.2$, $F_{\alpha,\beta}(x)$ for $p = 0.2 > F_{\alpha,\beta}(x)$ for $p = 0.4$

$$> F_{\alpha,\beta}(x) \text{ for } p = 0.6$$
$$> F_{\alpha,\beta}(x) \text{ for } p = 0.8.$$

It is seen that the limiting distribution function $F_{\alpha,\beta}(x)$ converges more quickly if more customers choose to take a bus (see Figure 3.6).

From (i) and (ii), customers should drive on 2-paths unless fewer customers generated from node α. From (iii), customers should take a bus. Therefore, the optimal policy is when $p = 0$.

At node u, the 1-path from node u to node β has a bus passing. If a customer chooses to drive on a 2-path, it is possible that a customer driving on a 2-path that has no bus passing or driving on a 2-path that has no bus passing on 2-1 path but has a bus passing on 2-2 path. So the traffic intensity on links of 2-path is smaller than that on the 1-path at the node u. We observe that:

(i) It may have advantages to choose a 2-path when the number of customers generated from the node u is not small. For example, when $p = 0.6$, the limiting distribution function $F_{u,\beta}(x)$ for $m_1 = 0.2$ is better than that for $m_1 = 0.8$ if $x > 9$. Moreover, for $p = 1$, the limiting distribution function $F_{u,\beta}(x)$ for $m_1 = 0.2$ is better than that for $m_1 = 1$ if $x > 6$ (see Figure 3.7 – 3.8).

(ii) When $m_1 = 0.2$, $F_{u,\beta}(x)$ for $p = 1 \; < F_{u,\beta}(x)$ for $p = 0.8$ if $x > 9$,

$F_{u,\beta}(x)$ for $p = 0.8 < F_{u,\beta}(x)$ for $p = 0.6$ if $x > 10$,

$F_{u,\beta}(x)$ for $p = 0.6 < F_{u,\beta}(x)$ for $p = 0.4$ if $x > 3$, and

$F_{u,\beta}(x)$ for $p = 0.4 < F_{u,\beta}(x)$ for $p = 0.2$ if $x > 1$.

These imply that as the probability that a customer chooses to drive is decreasing, the limiting distribution function $F_{u,\beta}(x)$ converges more quickly (see Figure 3.9 – 3.12).

From (i), customers should choose a 2-path generally. From (ii), for customers who drive on 2-paths with large probability, the limiting distribution function $F_{u,\beta}(x)$ converges quickly if the probability q that customers choose to take a bus increasing, therefore, the optimal policy is $p = 0$.

Figure 3.4: $K_1 = 10$, node $\alpha, (p, m_1) = (1, 0.2), (0.8, 0.2), (1, 1)$

Figure 3.5: $K_1 = 10$, node α, $(p, m_1) = (1, 1), (0.6, 0.2), (0.4, 0.2)$

Figure 3.6: $K_1 = 10$, node α, $m_1 = 0.2$, $p = 1, 0.8, 0.6, 0.4, 0.2$

Figure 3.7: $K_1 = 10$, node u, $(p, m_1) = (0.6, 0.2)$, $(0.6, 0.8)$

Figure 3.8: $K_1 = 10$, node u, $(p, m_1) = (1, 1)$, $(1, 0.2)$

Figure 3.9: $K_1 = 10$, node u, $(p, m_1) = (1, 0.2)$, $(0.8, 0.2)$

Figure 3.10: $K_1 = 10$, node u, $(p, m_1) = (0.8, 0.2)$, $(0.6, 0.2)$

Figure 3.11: $K_1 = 10$, node u, $(p, m_1) = (0.6, 0.2)$, $(0.4, 0.2)$

Figure 3.12: $K_1 = 10$, node u, $(p, m_1) = (0.2, 0.2)$, $(0.4, 0.2)$

(III) $K_1 = 5$

Each 2-path from node α to node β has a bus passing. The total arrival rate on each 2-path is larger for $K_1 = 5$ than that for $K_1 = 10$ or $K_1 = 18$. We observe that:

(i) When the probability that a customer takes a bus increases, the total arrival rate on links of 2-path will decrease, therefore, customers will have advantages to choose 2-paths. For example, when $m_1 = 0.2$, the limiting distribution function for $p < 1$ converges quickly than that for $p = 1$ (see Figure 3.13 – 3.14).

(ii) If all customers drive cars ($p = 1$), then the limiting distribution function will converge more slowly if more customers drive on 2-paths. As an example, Figure 3.15 shows that:

$F_{\alpha,\beta}(x)$ for $m_1 = 0.2 < F_{\alpha,\beta}(x)$ for $m_1 = 0.4$, and

$F_{\alpha,\beta}(x)$ for $m_1 = 0.4 < F_{\alpha,\beta}(x)$ for $m_1 = 1$.

(iii) As the probability of a customer chooses to take a bus increases, the total arrival rate on links of a 2-path will decrease. Since the waiting time for a bus is 0, customers who take a bus are as quickly as a customer drives on a 2-path. Therefore, if customers all take a bus, the traffic intensity will reduce and the end-to-end delay time will also reduce.

At node u, if a customer chooses to drive a car on a 2-path, customers either driving on a 2-path that has no bus passing or driving on a 2-path that has a bus passing on the 2-2 path. The waiting time for $K_1 = 5$ is smaller than that for $K_1 = 10$ or $K_1 = 18$, therefore, the arrival rate for a bus is large. The link from the node u to the node β has a bus passing, the traffic intensity on this 1-path is then large. We observe that:

(i) It may have advantages to choose a 2-path than the 1-path.

(ii) When $p = 1$, more customers choose to drive on a 2-path, the limiting distribution function $F_{u,\beta}(x)$ will converge more quickly.

(iii) If the number of customers generated from the node u is not large, it is better to choose a 1-path (see Figure 3.16).

(iv) If most of the customers drive by themselves, for example, $F_{u,\beta}(x)$ for $p = 0.8$ or 0.6 ($m_1 = 0.2$) is better than that for $p = 1$ ($m_1 = 0.2$) only if x is large. As the probability p decreases, $F_{u,\beta}(x)$ will increase. As an example, $F_{u,\beta}(x)$ for $p = 0.2$ ($m_1 = 0.2$) $> F_{u,\beta}(x)$ for $p = 1$ ($m_1 = 0.2$) if $x \geq 2$ (see Figure 3.17). As the probability p decreases, $F_{u,\beta}(x)$ converges more quickly.

From (i), (ii), (iii), and (iv), customers should drive on a 2-path rather than on a 1-path, and customers choose to take a bus will have advantages than to drive on 2-paths. Therefore, the optimal policy is $p = 0$

Figure3.13: $K_1 = 5$, node α, $m_1 = 0.2$, $p = 1, 0.8, 0.6, 0.4, 0.2$

Figure3.14: $K_1 = 5$, node α, $(p, m_1) = $ (1,1),$(0.8, 0.2), (0.6, 0.2), (0.4, 0.2)$

Figure3.15: $K_1 = 5$, node α, $(p, m_1) = (1, 1), (1, 0.4), (1, 0.2)$

Figure 3.16: $K_1 = 5$, node u, $(p, m_1) = (1, 1), (1, 0.2), (0.2, 0.2), (0.6, 0.2)$

Figure 3.17: $K_1 = 5$, node u, $(p, m_1) = (1, 0.2), (0.8, 0.2), (0.6, 0.2), (0.2, 0.2), (0, 0)$

4 Optimal Policies for a Network with Finite Node and Infinite Capacity

Assume that the capacity of a bus is infinite and the network has finite number of nodes. From equation (2.1), we know that the arrival rate with k customers on a link which has no bus passing is $\lambda_{k,N} = \sum_{i=k}^{m} b(i)\binom{i}{k}\ p^k[m_1^k(1-pm_1)^{i-k} + 2m_2(m_{2,N})^{m-1}(1-pm_{2,N})^{i-k}]$ v. When the number of nodes of the network decreases, the arrival rate will increase (*i.e.*, $\lambda_{k,N}$ becomes larger as the term $m_{2,N} = \frac{1-m_1}{2N-2}$ of $\lambda_{k,N}$ becomes larger). Figure 4.1 – 4.4 ($v = 0.05$, $T = 1$) provide evidences that $F_{\alpha,\beta}(x, N) \leq F_{\alpha,\beta}(x, N')$ if $N < N'$. It is clear that as the number of nodes of the network decreases, the probability that two customers driving on the same 2-path ($m_2 m_{2,N} = m_2 \frac{1-m_1}{2N-2}$) will increase. It then will take more travel time due to more customers joining the same 2-path queue. Hence q and m_2 will decrease as N decreases. If the number of nodes of the network is finite, we know that the last car of the same source joining the queue with length k has to wait for at least $k-1$ time units before departure. If the number of nodes of the network is infinite, the car on each 2-path may have zero waiting time. When $p = 1$ and $m_1 = 1$, $F_{\alpha,\beta}(x)$ and $F_{u,\beta}(x)$ are independent on N.

When the network has finite number of nodes, the limiting distribution functions (1.3) and (1.4) are no longer applicable, we will add some conditions to build the limiting distribution functions (1.3) and (1.4).

At node α, suppose that customers who take a bus traverse on a specific 2-path ($\alpha \to u \to \beta$). As the number of nodes decreases in the network, the probability that customers drive on the same 2-path increases. The probability that a specific customer drives on a specific 2-path is $m_{2,N}$ and the transmission time is $2T$. The probability that two specific customers drive on a specific 2-path is $(m_{2,N})^2$ and the transmission time is $3T$. In general, the probability that k specific customers drive on a specific 2-path is $(m_{2,N})^k$ and the transmission time is $(k+1)T$.

Let A_k be the probability that k customers drive on a specific 2-path such that the traveling time is less than or equal to x. Then $A_1 = m_{2,N}\ P(W' + W'' + 2T \leq x)$, $A_2 = (m_{2,N})^2\ P(W' + W'' + 3T \leq x)$, in general, $A_k = (m_{2,N})^k\ P(W' + W'' + (k+1)T \leq x)$.

Let B_k be the probability that k specific customers drive on 2-paths such that the total traveling times are less than or equal to x. Since there are $2N+1$ nodes in the network, every customer chooses a 2-path independently from its source to its destination. The total number of 2-paths is $2N-2$ for each different pair of source and destination. Assume that any node may generated at most 10 customers at any instant and the number of nodes of the network is large than 12. Then

$$B_1 = P_1^{2N-2} A_1,$$

$$B_2 = P_2^{2N-2}(A_1)^2 + P_1^{2N-2} A_2,$$

$$B_3 = P_3^{2N-2}(A_1)^3 + P_3^{2N-2} C_1^3 A_2 A_1 + P_1^{2N-2} A_3,$$

$$B_4 = P_4^{2N-2}(A_1)^4 + P_2^{2N-2}\frac{C_2^4 C_2^2}{2!}(A_2)^2 + P_3^{2N-2}\frac{C_2^4 C_1^2 C_1^1}{2!} A_2 (A_1)^2 + P_2^{2N-2} C_3^4 A_3 A_1 + P_1^{2N-2} A_4,$$

$$B_5 = P_5^{2N-2}(A_1)^5 + P_3^{2N-2}\frac{C_1^5 C_2^4 C_2^2}{2!} A_1 (A_2)^2 + P_3^{2N-2}\frac{C_3^5 C_1^2 C_1^1}{2!} A_3 (A_1)^2 + P_2^{2N-2} C_3^5 C_2^2 A_3 A_2 + P_2^{2N-2} C_1^5 A_4 A_1 + P_1^{2N-2} A_5,$$

$$B_6 = P_6^{2N-2}(A_1)^6 + P_3^{2N-2}\frac{C_2^6 C_2^4 C_2^2}{3!}(A_2)^3 + P_4^{2N-2}\frac{C_3^6 C_1^3 C_1^2 C_1^1}{3!} A_3 (A_1)^3 + P_3^{2N-2} C_3^6 C_2^3 C_1^1 A_3 A_2 A_1 + P_2^{2N-2}\frac{C_3^6 C_3^3}{2!}(A_3)^2 + P_3^{2N-2}\frac{C_4^6 C_1^2 C_1^1}{2!} A_4 (A_1)^2 + P_2^{2N-2} C_4^6 C_2^2 A_4 A_2 + P_3^{2N-2}\frac{C_2^6 C_2^4}{2!}(A_1)^2 (A_2)^2 + P_2^{2N-2} C_5^6 C_1^1 A_5 A_1 + P_1^{2N-2} A_6,$$

.

.

$$B_{10} =$$

$$P_{10}^{2N-2}(A_1)^{10} + P_9^{2N-2} C_2^{10} A_2 (A_1)^8 + P_8^{2N-2}\frac{C_2^{10} C_2^8}{2!}(A_2)^2 (A_1)^6 + P_7^{2N-2}\frac{C_2^{10} C_2^8 C_2^6}{3!}(A_2)^3 (A_1)^4 +$$
$$P_6^{2N-2}\frac{C_2^{10} C_2^8 C_2^6 C_2^4}{4!}(A_2)^4 (A_1)^2 + P_5^{2N-2}\frac{C_2^{10} C_2^8 C_2^6 C_2^4 C_2^2}{5!}(A_2)^5 + P_8^{2N-2}\frac{C_3^{10} C_1^7 C_1^6 C_1^5 C_1^4 C_1^3 C_1^2 C_1^1}{7!} A_3 (A_1)^7 +$$
$$P_7^{2N-2} C_3^{10} C_2^7 A_3 A_2 (A_1)^5 + P_6^{2N-2}\frac{C_3^{10} C_2^7 C_2^5 C_1^3 C_1^2 C_1^1}{2!3!} A_3 (A_2)^2 (A_1)^3 + P_5^{2N-2}\frac{C_3^{10} C_2^7 C_2^5 C_2^3}{3!} A_3 (A_2)^3 A_1 +$$
$$P_6^{2N-2}\frac{C_3^{10} C_3^7 C_1^4 C_1^3 C_1^2 C_1^1}{2!4!}(A_3)^2 (A_1)^4 + P_5^{2N-2}\frac{C_3^{10} C_3^7 C_2^4}{2!}(A_3)^2 A_2 (A_1)^2 + P_4^{2N-2}\frac{C_3^{10} C_3^7 C_2^4 C_2^2}{2!2!}(A_3)^2 (A_2)^2 +$$
$$P_4^{2N-2}\frac{C_3^{10} C_3^7 C_3^4 C_1^1}{3!}(A_3)^3 A_1 + P_7^{2N-2}\frac{C_4^{10} C_1^6 C_1^5 C_1^4 C_1^3 C_1^2 C_1^1}{6!} A_4 (A_1)^6 + P_5^{2N-2}\frac{C_4^{10} C_2^6 C_1^4 C_1^3 C_1^2 C_1^1}{4!} A_4 A_2 (A_1)^4 +$$
$$P_5^{2N-2}\frac{C_4^{10} C_2^6 C_2^4 C_1^2 C_1^1}{2!2!} A_4 (A_2)^2 (A_1)^2 + P_4^{2N-2}\frac{C_4^{10} C_2^6 C_2^4 C_2^2}{3!} A_4 (A_2)^3 + P_5^{2N-2}\frac{C_4^{10} C_3^6 C_1^3 C_1^2 C_1^1}{3!} A_4 A_3 (A_1)^3 +$$
$$P_4^{2N-2} C_4^{10} C_3^6 C_2^3 C_1^1 A_4 A_3 A_2 A_1 + P_3^{2N-2}\frac{C_4^{10} C_3^6 C_3^3}{2!} A_4 (A_3)^2 + P_4^{2N-2}\frac{C_4^{10} C_4^6 C_1^2 C_1^1}{2!2!}(A_4)^2 (A_1)^2 +$$
$$P_3^{2N-2}\frac{C_4^{10} C_4^6 C_2^2}{2!}(A_4)^2 A_2 + P_6^{2N-2}\frac{C_5^{10} C_1^5 C_1^4 C_1^3 C_1^2 C_1^1}{5!} A_5 (A_1)^5 + P_4^{2N-2}\frac{C_5^{10} C_2^5 C_2^3 C_1^1}{2!} A_5 (A_2)^2 A_1 +$$
$$P_5^{2N-2}\frac{C_5^{10} C_2^5 C_1^3 C_1^2 C_1^1}{3!} A_5 A_2 (A_1)^3 + P_3^{2N-2} C_5^{10} C_3^5 C_2^2 A_5 A_3 A_2 + P_4^{2N-2}\frac{C_5^{10} C_3^5 C_1^2 C_1^1}{2!} A_5 A_3 (A_1)^2 +$$
$$P_3^{2N-2} C_5^{10} C_4^5 A_5 A_4 A_1 + P_2^{2N-2}\frac{C_5^{10} C_5^5}{2!}(A_5)^2 + P_5^{2N-2} C_6^{10} A_6 (A_1)^4 + P_4^{2N-2} C_6^{10} C_2^4 A_6 A_2 (A_1)^2 +$$
$$P_3^{2N-2}\frac{C_6^{10} C_2^4 C_2^2}{2!} A_6 (A_2)^2 + P_3^{2N-2} C_6^{10} C_3^4 C_1^1 A_6 A_3 A_1 + P_2^{2N-2} C_6^{10} C_4^4 A_6 A_4 + P_4^{2N-2} C_7^{10} A_7 (A_1)^3 +$$
$$P_3^{2N-2} C_7^{10} C_2^3 A_7 A_2 A_1 + P_2^{2N-2} C_7^{10} A_7 A_3 + P_3^{2N-2} C_8^{10} A_8 (A_1)^2 + P_2^{2N-2} C_8^{10} A_8 A_2 + P_2^{2N-2} C_9^{10} A_9 A_1$$

$+P_1^{2N-2}A_{10}$.

In equations B_i, $i = 1, 2, 3, ..., 10$, we have considered all the possibilities that customers drive on all 2-paths when the number of nodes of the network is finite. The limiting distribution functions (1.3) and (1.4) can also be rewritten as:

$$\begin{aligned} F_{\alpha,\beta}(x, N) &= \textstyle\sum_i b(i)\ p^i[m_1^i P(W{+}iT{\leq}x){+}\sum_{j=1}^{i}\binom{i}{j} m_1^{i-j} m_2^j P(W{+}(i\text{-}j)T{\leq}x)B_j] \\ &+ \textstyle\sum_{k=1}^{m-1}\sum_{i=k+1}^{m} b(i)\binom{i}{k} p^{i-k}q^k[m_1^{i-k}P(W{+}(i\text{-}k)T{\leq}x){+} \\ &\textstyle\sum_{j=1}^{i-k}\binom{i-k}{j} m_1^{i-k-j} m_2^j P(W{+}(i\text{-}k\text{-}j)T{\leq}x)B_j]P(W'{+}W''{+}W_\alpha{+}2T{\leq}x) \\ &+ \textstyle\sum_i b(i) q^i P(W'{+}W''{+}W_\alpha{+}2T \leq x), \text{ and} \end{aligned} \quad (4.1)$$

$$\begin{aligned} F_{u,\beta}(x, N) &= \textstyle\sum_i b(i)\ p^i[m_1^i P(W{+}iT{\leq}x){+}\sum_{j=1}^{i}\binom{i}{j} m_1^{i-j} m_2^j P(W{+}(i\text{-}j)T{\leq}x)B_j] \\ &+ \textstyle\sum_{k=1}^{m-1}\sum_{i=k+1}^{m} b(i)\binom{i}{k} p^{i-k}q^k[m_1^{i-k}P(W{+}(i\text{-}k)T{\leq}x){+} \\ &\textstyle\sum_{j=1}^{i-k}\binom{i-k}{j} m_1^{i-k-j} m_2^j P(W{+}(i\text{-}k\text{-}j)T{\leq}x)B_j]P(W''{+}W_u{+}T{\leq}x) \\ &+ \textstyle\sum_i b(i) q^i P(W''{+}W_u{+}T{\leq}x). \end{aligned} \quad (4.2)$$

In Figure 4.1 – 4.4, we find that as the number of the nodes decreases, the distribution functions (4.1) and (4.2) converge slowly than the limiting case. The above phenomenon is more obvious when the value of m_2 is large. This implies that customers should drive on a 1-path rather than a 2-path.

Figre 4.1: $K_1 = 5$, node α, $(p, m_1) = (0.4, 0.2)$

Figre 4.2: $K_1 = 5$, node α, $(p, m_1) = (0.8, 0.2)$

Figre 4.3: $K_1 = 18$, node α, $(p, m_1) = (1, 0.2)$

Figre 4.4: $K_1 = 10$, node u, $(p, m_1) = (0.4, 0.2)$

5 Optimal Policies for a Network with Unlimited Node and Finite Capacity

Consider the situation that the network has infinite node and the capacity of a bus is finite. We will discuss the case $K_1 = 3$ and the transmission time T is 1 at the node u. At node α, the number of 2-paths is infinite, there is always a bus available when customers generated. So there will be no affect if the capacity of a bus is reduced and the limiting distribution function $F_{\alpha,\beta}(x)$ will be the same as the case when the capacity of a bus is infinite. At the node u, there is a bus scheduled every three minutes and customers that take a bus need to wait until all customers that generated from the node u preceded them get on to the bus. Let $Z_k = \sum_{i=k}^{m} b(i)\binom{i}{k}q^k p^{i-k}v$ be the probability that k customers generated from the node u in one minute and wait to take a bus. Let E_k be the probability that there are k seats available on a bus. Suppose that the capacity of a bus is $2m$, we obtain $E_0 = (Z_m)^2$, $E_1 = 2Z_{m-1}Z_m$, $E_2 = 2Z_mZ_{m-2} + (Z_{m-1})^2$, $E_3 = 2(Z_mZ_{m-3} + Z_{m-1}Z_{m-2})$, $E_4 = 2(Z_{m-1}Z_{m-3}+Z_mZ_{m-4})+(Z_{m-2})^2$, and $E_5 = 2(Z_mZ_{m-5}+Z_{m-1}Z_{m-4}+Z_{m-2}Z_{m-3})$. In general, $E_k = \sum_{a=0}^{k} Z_{m-a}Z_{m-k+a}$ be the probability that there are k seats available on a bus. The limiting distribution function (1.4) can be easily derive as:

$$
\begin{aligned}
F_{u,\beta}(x) = & \sum_{i=1}^{m} b(i)\ p^i[m_1^i P(W+i\leq x) + \sum_{j=1}^{i} \binom{i}{j} m_1^{i-j} m_2^j P(W+(i-j)\leq x) P(W'+W''+2\leq x)^j] \\
& + \sum_{k=1}^{m-1} \sum_{i=k+1}^{m} b(i)\binom{i}{k} p^{i-k} q^k [m_1^{i-k} P(W+(i-k)\leq x) + \sum_{j=1}^{i-k} \binom{i-k}{j} m_1^{i-k-j} m_2^j P(W+(i-k-j)\leq x) \\
& P(W'+W''+2\leq x)^j][P(W''+W_u+1\leq x)(1-\sum_{s=0}^{k-1} E_s) + P(W''+W_u+4\leq x)(\sum_{s=0}^{k-1} E_s)] \\
& + \sum_{i=1}^{m} b(i) q^i [P(W''+W_u+1\leq x)(1-\sum_{s=0}^{i-1} E_s) + P(W''+W_u+4\leq x)(\sum_{s=0}^{i-1} E_s)]. \qquad (5.1)
\end{aligned}
$$

As the capacity of a bus decreases, and the rate of customers generated from node u is large, customers who wait to take a bus may have no seats. So the limiting distribution function converges slowly than the case when the capacity of a bus is infinite (see Figure 5.1 – 5.2).

Figure 5.1: $v = 1, K_1 = 3$, node u, $p = 0$

Figure 5.2: $v = 0.8$, $K_1 = 3$, node u, $(p, m_1) = (0.2, 1)$

6 Conclusions

From the discussions in the previous sections, we know that if there are less customers generated from a node and the waiting time of a bus is not small (for example $K_1 = 18$) , it is generally better for a customer to drive on a 1-path than take a bus or drive on a 2-path. For $K_1 = 5$ or $K_1 = 10$, if the capacity of a bus is infinite, the optimal policy is $p = 0$ (all customers take a bus). In these two cases the waiting time for a bus is small. But if the waiting time of a bus is large (for example $K_1 = 18$), we will choose to drive rather than to take a bus. It is also clear that as the probabilities of customers taking a bus increase, the traffic intensity on 2-paths would decrease. Therefore, customers that choose to drive would have advantages to drive on a 2-path. When the number of nodes of network decreases, the probability of a customer drive on a 2-path should decrease. So the optimal policy will tend to $p = 1$ (if the waiting time for a bus is large) or $p = 0$ (if the waiting time for a bus is small). It is obvious that as the capacity of a bus decreases, the probability of a customer taking a bus would also decrease. In reality, customers should not drive if it is more convenience to take a public transportation. In this model, we assume that the transmission time T is fixed and ignore the distance between each link $\overrightarrow{ij}$. It should incorporate considerations of the distance function or variable transmission time in the model further to simulate the real situations.

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