The Optimum Plan and Accurate Inference for Accelerated Life Test under Inverse Gaussian Distribution

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ABSTRACT

Industry develops very quickly in recent years. High quality products are requested with high reliability. So how to use methods in short time to estimate the life of object is interested. Accelerated life test is a practical method, which place product into a test in some special (accelerated) state by characters, for example, with high temperature and voltage. We obtain information in accelerated conditions and estimate exact life time in the normal operating conditions. This method does not only save time but also save cash.

Here we consider the optimum accelerated plan that employs the minimization of standard deviation for estimation to find the proper low and high stress for testing. The best compromise plan is a design that minimize the standard deviation of the target estimator with three stress in equal space and with same allocation of observations on the stress.

We derive the Fisher information and employ the large sample theory to obtain the standard deviation of $\widehat{t_p}$ (p-th quantile of the distribution). The method based on the normal approximation theory to obtain the stress and allocation of the plan may not result in the real optimum state when sample size is less than 1000. So we propose to find the optimum stress and allocation by simulation.

In this thesis, we overcome some predicament in generating Inverse Gaussian random variables. For the most common procedure, one uses uniform random variable and solve the inverse of CDF to obtain the required random variable. This is quite time consuming, so we use the method provided by Michael (1976) based on Chi-square distribution with degree of freedom 1.

It is known that the bias problem could be serious when censoring is presented. We use the bias correction by applying bootstrap procedure to correct the estimate of $\widehat{t_p}$.

1 Introduction

1.1 Motivation

In the time of high technology, high quality products usually don't fail in short time usage; for example, cellular phone, integrated circuit, light bulb, etc. In this study, the primary problem of interest is how to detect exact life of the objects in a small sample and under the restricted time.

Accelerated Life Test is a common way to look foward to obtain life of products in the very short time. The optimum design considered in the literature is typically optimum in the sense of minimization over the asymptotic variance. But in real cases, the design obtained by this way may not be optimum. Monte Carlo simulation is a method to deal with the finite-sample. To find out the finite sample property, we are looking for the optimum design by simulation. Anticipate that acccurate result will be obtained in the smaller sample case.

Due to some qualities of commodity, the life time of target objects may follow Inverse Gaussian distribution. We focus on this distribution. The typical form of censored data includes both Type I (time) and Type II (failure). Because Type I censoring in the testing is close to the practical needs, we consider this censoring mechanism in this paper. Type I censoring means that data are observed until some specific time (people decide on this time) and censor after the time. It is known that the estimation under censoring could cause bias. Bias correction is a value to amend value of estimation. We use parameteric bootstrap to give the bias correction.

1.2 Literature Review

Menzefricke (1992) discussed sample size planning for accelerated life tests in the linear regression model when there is type II censoring. Adcock (1997) concerned with methods of sample size determination, the methods covered are in two groups: frequentist and Bayesian. Escobar and Meeker (1998) showed how to compute the

Fisher information matrix and the asymptotic covariance matrix for maximum likelihood estimators for a wide class of parametric models. Jeng and Meeker (2000) used Monte Carlo simulation to investigate the finite-sample properties of some procedures.

1.2.1 IG Distribution

Tweedie (1947) explained the inverse relationship for the cumulant generating functions between Gaussian and Inverse Gaussian (IG) distribution. Michael, Schucany and Haas (1976) generated random variates of IG distribution using transformations with multiple roots, from the chi-square distribution with 1 degree of freedom. Chhikara and Folks (1977) showed that failure rate of IG distribution is nonmonotonic, initially increasing and then decreasing. Whitmore and Yalovsky (1978) presented a logarithmic transformation for IG variates which produces approximate normality for large values of the concentration parameter. The book by Chhikara and Folks (1989) described the IG distribution in detail. The IG distribution has its spring in the Brownian motion as a first passage time distribution. We obtain it in view of a Wiener process. Meeker and Escobar (1998, pp104-105) described the properties of IG distribution and gave a figure of CDF, pdf and hazard function. Onar and Padgett (2000) presented a continuous damage model based on a Gaussian process, a family of Inverse Gaussian accelerated test models is obtained.

1.2.2 Accelerated Life Test

Nelson and Kielpinski (1976) suggested that the optimum plan should use two test stresses and more units at the lower stress. They reparametrized model by converting the stress of quantity into absolute stress between 0 and 1. Nelson and Meeker (1978) presented maximum likelihood theory for large-sample optimum accelerated life test plans. Meeker (1984) considered the situation that the criteria for comparing test plans was chosen to minimize the asymptotic variance. Furthermore, they discussed the departures from the assumed stress-life relationship and robustness to departures from the assumptions used in determining the plans. Nelson (1990) described the accelerated life test plans in his book. The Arrhenius and inverse power relationships are usually used for life-stress relationships. Those sort of models can simplify the complex life-stress relationships. Doksum and Hóyland (1992) use Wiener-process approach an alternative flexible class of time-transformed IG models in which time to failure is modeled in terms of accumulated decay reaching level and in which parametric functions are used to express how higher stresses accelerate the rate of decay and the time to failure.

1.2.3 Bootstrap and Bias Correction

Usually one assumes that error of estimation in the early period of a design plan could be large especially when the sample size is not large enough. Also point estimate under the situation that data are censored would cause some bias. For these reasons one takes a correction for the evaluation is important. When a real-valued parameter θ is interested, we will find an estimator called $\widehat{\theta}$. The bias of $\widehat{\theta}$ ($\widehat{\theta}$ as an estimate of θ) is defined as $bias = E\left[\widehat{\theta}\right] - \theta$. The bias-corrected estimator is defined as $\widehat{\theta} - \widehat{bias}$. Efron and Tibshirani (1993, Chapter 6) described the bootstrap procedure in detail, including sampling process and the parametric bootstrap. Meeker and Escobar (1998, Chapter 9) explained the methods for generating bootstrap samples and for obtaining the bootstrap confidence intervals for life data. The idea of bootstrap sampling is to simulate the repeated sampling and use the message from the distribution of appropriate statistics to reduce the reliance on large-sample approximations.

1.3 Overview

The remainder of this paper is organized as follows. Section 2 describes properties of model of IG distribution and discusses the criteria of the accelerated life test plans. Section 3 shows how to use delta method to obtain distributions of quantile estimator by asymptotic normal theory. Section 4 discusses the sample size problem for obtaining optimum plans. Section 5 generate IG distribution from Chi-square distribution with degree of freedom 1, and uses simulation method to compare with results from asymptotic normal theory. Section 6 uses bootstrap method to estimate bias and to give bias correction. Section 7 describes main results. Section 8 suggests some directions for future research.

2 Model

2.1 Wiener process and Inverse Gaussian Distribution

This section presents the distribution of Inverse Gaussian (IG), which is derived from the Brownian motion as a first passage time distribution. The Brownian motion process is sometimes called the Wiener process. There are two parameters, drift ν and diffusion σ^2. A Wiener process W(t) follows the properties below:

1. The non overlay differences of W(t) are independent. That is, when $0 \leq t_1 < t_2$ ... $t_i < t_{i+1} \leq \infty$, $W(t_2) - W(t_1)$, ..., $W(t_{i+1}) - W(t_i)$ are independent.

2. $W(t_{i+1}) - W(t_i) \sim$ N $(\nu(t_{i+1} - t_i), \sigma^2(t_{i+1} - t_i))$, for $t_{i+1} > t_i$.

If $\nu = 0$ and $\sigma = 1$, the process is called the standard Brownian motion B_t (Ross 1983, Chapter 6). The form of Wiener process can be obtained as $W(t) = \nu t + \sigma B_t$.

When the process reaches a critical quality, the first passage time T follows distribution of IG (Chhikara and Folks, 1989, Chapter 3). Now we describe the arguments to obtain the relationship between (ν, σ^2) and (μ, ϕ) where (μ, ϕ) are the parameters of IG. Let $Q(t|s_0)$ be the quality of detected object in the stress s_0 which may be temperature or voltage or others. Let D_Q denote the lower critical value of the quality. When Q reaches D_Q, the product could be claimed as failed. Suppose we can find a transformable function φ by character of product, such that $\varphi(Q(t|s_0))$ satisfy the Wiener process with drift ν and σ^2. So the first passage time T is defined as $T = \inf\{t|\varphi\left(Q(t|s_0)\right) \leq \varphi\left(D_Q\right)\}$.

The form of CDF of IG distribution is:

$$F(t;\mu,\phi) = \Phi\left[\sqrt{\frac{\mu\phi}{t}}(\frac{t}{\mu}-1)\right] + e^{2\phi}\Phi\left[-\sqrt{\frac{\mu\phi}{t}}(\frac{t}{\mu}+1)\right]$$

where $\Phi(\cdot)$ is the CDF of standard normal.

The pdf of IG distribution is:

$$f(t;\mu,\phi) = \sqrt{\frac{\mu\phi}{2\pi}}t^{-\frac{3}{2}}\exp\left[-\frac{\phi(t-\mu)^2}{2\mu t}\right], \quad for\ all \quad t > 0.$$

where $\mu = \varphi(D_Q)/\nu$ and $\phi = \nu\varphi(D_Q)/\sigma^2$.

The character of product $Q(t|s_0)$ often is not expressed in numeric form or the primitive data are hard to analysis. It is important for analysis to identify the dataset which determined by a professional with specialized knowledge in order to obtain φ.

Suppose $Q(t|s)$ be the quality of product in the stress s, denote ξ as a transformation between s_0 and s. Consider:

$$Q(t|s) = Q(e^{A\xi}t|s_0)$$

, hence

$$\begin{aligned}\varphi(Q(t|s)) &= \nu(e^{A\xi}t) + \sigma B_{e^{A\xi}t} \\ &= (e^{A\xi}\nu)t + (e^{\frac{A\xi}{2}}\sigma)B_t\end{aligned}$$

where $A > 0$, then $\varphi((Q(t|s))$ be a Wiener process with drift $e^{A\xi}\nu$ and diffusion $e^{A\xi}\sigma^2$. As defined above we have random variates following $IG(\mu(\xi), \phi(\xi))$ at the stress s, where

$$\begin{aligned}\mu(\xi) &= \frac{\varphi(D_Q)}{e^{A\xi}\nu} = \frac{\mu}{e^{A\xi}} \\ \phi(\xi) &= \frac{e^{A\xi}\nu\varphi(D_Q)}{e^{A\xi}\sigma^2} = \phi\end{aligned}$$

Take logarithm for first equation, and consider $\beta_0 = \ln(\mu)$; $\beta_1 = -A$, then we can obtain below:

$$\ln\mu(\xi) = \beta_0 + \beta_1\xi \tag{1}$$

By results above, we assume the criteria below:

1. The failure time of the every experiment units follow IG distribution with mean μ and shape ϕ.

2. There is a relation between the mean μ and the transformable function ξ such that

$$\ln \mu(\xi) = \beta_0 + \beta_1 \xi. \tag{2}$$

3. The shape ϕ is a constant.

4. In this thesis we choose a pre-time η and the experiment runs until time η. That is an experiment with Type I censoring.

2.2 Accelerated Life Test Plan and Criteria

Experimental parameters. For two stress design, let ξ_H and ξ_L denote the high and low stress level respectively, and n the sample of units, $n = n_{\xi_H} + n_{\xi_L}$ (n_{ξ_H} be number of test units at the high stress, n_{ξ_L} be number of test units used at the low stress). Let π_L be the proportion of total test units used at the low stress, thus we define $\pi_L = n_{\xi_L}/n$. Then $\pi_L + \pi_H = 1$. For three stress design, let ξ_M denote the median stress level. Then $n = n_{\xi_H} + n_{\xi_M} + n_{\xi_L}$, $\pi_L + \pi_M + \pi_H = 1$.

Criteria. In an accelerated life test the model parameters are estimated under the experiment conditions. The standard deviation of p-th quantile SD_{t_p} can be estimated by using the estimator of the model parameters. The criteria of the optimum in this paper is the test plan that minimize the standard deviation of $\widehat{t_p}$.

Best Standard Plans. There are K equally spaced test stresses and any allocation of observations for test stress levels is equal to the allocation of other level. One obtains best standard plans by choosing ξ_L to minimize $SD_{\widehat{t_p}}$.

Optimum Plans. Consider a plan that has only two stress levels in the experiment which are ξ_L and ξ_H. The plan is optimum in the sense that $\widetilde{\xi_L}$ and $\widetilde{\xi_H}$ give the smallest standard deviation of the estimator (e.g., $\widehat{t_p}$) among all possible ξ_L and ξ_H.

Best Compromise Plans. The Best Compromise Plans have equal allocation of observations on the three levels of stress and have the restriction that ξ_M is the average of ξ_L and ξ_H. If the accelerated relation is not linear, the plan also can catch the quadratic curvature. The test plan is considered with respect to nonlinear investigation.

2.3 Reparametrization (Standardized Stress)

Let x_D denote a product designed stress (normal stress), that is, product on the routine state. The x is different from the distinct test plans, therefore, it results in different scale by the measure needed. By standardization, let

$$\xi = \xi(x) = \frac{x - x_D}{x_H - x_D}.$$

Then the normal stress ξ_D=0 and the highest stress ξ_H=1. The level of design stresses are between 0 and 1, it is useful in making the calculation easier. The equation (1) can be changed and form below:

$$\ln \mu(\xi) = \beta_0 + \beta_1 \xi$$

where

$$\beta_0 = \alpha_0 + \alpha_1 x_D$$

$$\beta_1 = \alpha_1 (x_H - x_D)$$

3 Asymptotic Normality

3.1 Likelihood Function and Fisher Information Matrix

Suppose the failed time T follow the Inverse Gaussian distribution with parameter $\mu(\xi_i)$ and ϕ respectively at the stress ξ_i for $i = 1, 2, ..., K$. Let T_{ij} denote the jth observation at stress ξ_i. There are n units in the design plan and $n = n_1+n_2+...+n_K$ for n_i be the number of units at the stress ξ_i, the allocation of observation for ξ_i is $\pi_i = n_i/n$. The form of likelihood function of IG with Type I censoring is given below:

Let δ_{ij} denote the indicator function, η be the censoring time.

$$\delta_{ij} = \begin{cases} 1, & T_{ij} \leq \eta, \\ 0, & T_{ij} > \eta. \end{cases}$$

Log Likelihood Function of IG:

$$\begin{aligned} L_{ij} =& \delta_{ij} \left\{ \frac{1}{2} \left[-\ln(2\pi) + \beta_0 + \beta_1 \xi_i + \ln \phi \right] - \frac{3}{2} \ln t_{ij} - \frac{\phi \left(t_{ij} - e^{\beta_0+\beta_1\xi_i} \right)^2}{2e^{\beta_0+\beta_1\xi_i} t_{ij}} \right\} \\ & + (1-\delta_{ij}) \ln \left[1 - F_{\xi_i}(\eta) \right] \\ & \textit{where } T_{ij} \sim IG(e^{\beta_0+\beta_1\xi_i}, \phi),\ F_{\xi_i}(\eta) \textit{ is CDF of IG at censor time } \eta. \end{aligned}$$

Fisher Information Matrix. Consider $\boldsymbol{\theta} = (\beta_0, \beta_1, \phi)$, first we calculate the first and second-order partial differential function for the parameter $\boldsymbol{\theta}$ about L_{ij}. Secondly we take expectation of the second-order partial differential function, then the Fisher information matrix $I_{ij}(\boldsymbol{\theta})$ can be expressed as

$$I_{ij}(\boldsymbol{\theta}) = \mathrm{E} \begin{bmatrix} -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \beta_0^2} & -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \beta_0 \partial \phi} \\ -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \beta_1 \partial \beta_0} & -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \beta_1^2} & -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \beta_1 \partial \phi} \\ -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \phi \partial \beta_0} & -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \phi \partial \beta_1} & -\frac{\partial^2 L_{ij}(\boldsymbol{\theta})}{\partial \phi^2} \end{bmatrix}_{3\times3}$$

where

$$
\begin{aligned}
E\left[-\frac{\partial}{\partial\beta_0}\left(\frac{\partial L_{ij}\left(\boldsymbol{\theta}\right)}{\partial\beta_0}\right)\right] =& \phi F_{\xi_i}(\eta)\left(\frac{1}{2\phi}+1\right)-\phi\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)+\frac{\partial^2 F_{\xi_i}(\eta)}{\partial\beta_0^2} \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\left[\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\right]^2 \\
E\left[-\frac{\partial}{\partial\beta_1}\left(\frac{\partial L_{ij}\left(\boldsymbol{\theta}\right)}{\partial\beta_1}\right)\right] =& \phi\xi_i^2 F_{\xi_i}(\eta)\left(\frac{1}{2\phi}+1\right)-\phi\xi_i^2\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)+\frac{\partial^2 F_{\xi_i}(\eta)}{\partial\beta_1^2} \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\left[\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_1}\right]^2 \\
E\left[-\frac{\partial}{\partial\phi}\left(\frac{\partial L_{ij}\left(\boldsymbol{\theta}\right)}{\partial\phi}\right)\right] =& \frac{1}{2\phi^2}F_{\xi_i}(\eta)+\frac{\partial^2}{\partial\phi^2}F_{\xi_i}(\eta)+\left[1-F_{\xi_i}(\eta)\right]^{-1}\left[\frac{\partial F_{\xi_i}(\eta)}{\partial\phi}\right]^2 \\
E\left[-\frac{\partial}{\partial\beta_1}\left(\frac{\partial L_{ij}\left(\boldsymbol{\theta}\right)}{\partial\beta_0}\right)\right] =& \phi\xi_i F_{\xi_i}(\eta)\left(\frac{1}{2\phi}+1\right)-\phi\xi_i\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)+\frac{\partial}{\partial\beta_1}\left(\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\right) \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_1} \\
E\left[-\frac{\partial}{\partial\phi}\left(\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial\beta_0}\right)\right] =& \frac{1}{2\phi}F_{\xi_i}(\eta)-\frac{1}{\phi}\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}+\frac{\partial}{\partial\phi}\left(\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\right) \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\frac{\partial F_{\xi_i}(\eta)}{\partial\phi} \\
E\left[-\frac{\partial}{\partial\phi}\left(\frac{\partial L_{ij}\left(\boldsymbol{\theta}\right)}{\partial\beta_1}\right)\right] =& \frac{\xi_i}{2\phi}F_{\xi_i}(\eta)-\frac{1}{\phi}\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_1}+\frac{\partial}{\partial\phi}\left(\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_1}\right) \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_1}\frac{\partial F_{\xi_i}(\eta)}{\partial\phi}
\end{aligned}
$$

Define $I\left(\boldsymbol{\theta}\right)=\sum_{i=1}^{k}\sum_{j=1}^{n_i}I_{ij}\left(\boldsymbol{\theta}\right)$, that is the Fisher information for all observations in the experiment. We give the calculation about the first and second-order partial differential for the $\boldsymbol{\theta}$ and expectation of the second-order partial differential function.

First order partial differential function for β_0.

$$\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_0} = \delta_{ij}\left[\frac{1}{2} - \frac{\phi}{2t_{ij}}\frac{-2\left(t_{ij} - e^{(\beta_0+\beta_1\xi_i)}\right)\left(e^{2(\beta_0+\beta_1\xi_i)}\right) - \left(t_{ij} - e^{(\beta_0+\beta_1\xi_i)}\right)^2\left(e^{(\beta_0+\beta_1\xi_i)}\right)}{e^{2(\beta_0+\beta_1\xi_i)}}\right]$$
$$+ (1-\delta_{ij})\left[\frac{-\frac{\partial}{\partial\beta_0}\left(F_{\xi_i}(\eta)\right)}{(1-F_{\xi_i}(\eta))}\right]$$
$$= \delta_{ij}\left[\frac{1}{2} - \frac{\phi}{2t_{ij}}\frac{2e^{(\beta_0+\beta_1\xi_i)} - t_{ij}^2}{e^{(\beta_0+\beta_1\xi_i)}}\right] + (1-\delta_{ij})\left[\frac{-\frac{\partial}{\partial\beta_0}\left(F_{\xi_i}(\eta)\right)}{(1-F_{\xi_i}(\eta))}\right]$$
$$= \delta_{ij}\left[\frac{1}{2} - \frac{\phi}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)} + \frac{\phi}{2}\frac{t_{ij}}{e^{(\beta_0+\beta_1\xi_i)}}\right] + (1-\delta_{ij})\left[\frac{-\frac{\partial}{\partial\beta_0}\left(F_{\xi_i}(\eta)\right)}{(1-F_{\xi_i}(\eta))}\right]$$

First order partial differential function for β_1.

$$\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_1} = \delta_{ij}\left[\frac{1}{2}\xi - \frac{\phi\xi}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)} + \frac{\phi\xi}{2}\frac{t_{ij}}{e^{(\beta_0+\beta_1\xi_i)}}\right] + (1-\delta_{ij})\left[\frac{-\frac{\partial}{\partial\beta_1}\left(F_{\xi_i}(\eta)\right)}{(1-F_{\xi_i}(\eta))}\right]$$

First order partial differential function for ϕ.

$$\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \phi} = \delta_{ij}\left[\frac{1}{2\phi} - \frac{t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}} + 1 - \frac{e^{(\beta_0+\beta_1\xi_i)}}{2t_{ij}}\right] + (1-\delta_{ij})\left[\frac{-\frac{\partial}{\partial\phi}\left(F_{\xi_i}(\eta)\right)}{(1-F_{\xi_i}(\eta))}\right]$$

Second order partial differential function for β_0.

$$\begin{aligned}\frac{\partial}{\partial\beta_0}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_0} =&\delta_{ij}\left[-\frac{\phi}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)} - \frac{\phi t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right] \\ &+ (1-\delta_{ij})\frac{\left[-\frac{\partial^2}{\partial\beta_0^2}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right) - \left(\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\right)^2\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\end{aligned} \tag{3}$$

Second order partial differential function for β_1.

$$\begin{aligned}\frac{\partial}{\partial\beta_1}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_1} =&\delta_{ij}\left[-\frac{\phi\xi^2}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)} - \frac{\phi\xi^2 t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right] \\ &+ (1-\delta_{ij})\frac{\left[-\frac{\partial^2}{\partial\beta_1^2}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right) - \left(\frac{\partial}{\partial\beta_1}F_{\xi_i}(\eta)\right)^2\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\end{aligned} \tag{4}$$

Second order partial differential function for ϕ.

$$\frac{\partial}{\partial\phi}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \phi} =\delta_{ij}\left(-\frac{1}{2\phi^2}\right) + (1-\delta_{ij})\frac{\left[-\frac{\partial^2}{\partial\phi^2}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right) - \left(\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\right)^2\right]}{\left(1-F_{\xi_i}(\eta)\right)^2} \tag{5}$$

Second order partial differential function for β_0 and β_1.

$$\begin{aligned}\frac{\partial}{\partial \beta_1}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_0} =&\delta_{ij}\left[-\frac{\phi\xi}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)}-\frac{\phi\xi t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right]\\ &+(1-\delta_{ij})\frac{\left[-\frac{\partial}{\partial\beta_1}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right)-\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\frac{\partial}{\partial\beta_1}F_{\xi_i}(\eta)\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\end{aligned} \quad (6)$$

Second order partial differential function for β_0 and ϕ.

$$\begin{aligned}\frac{\partial}{\partial \phi}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_0} =&\delta_{ij}\left[-\frac{1}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)}+\frac{t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right]\\ &+(1-\delta_{ij})\frac{\left[-\frac{\partial}{\partial\phi}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right)-\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\end{aligned} \quad (7)$$

Second order partial differential function for β_1 and ϕ.

$$\begin{aligned}\frac{\partial}{\partial \phi}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_1} =&\delta_{ij}\left[-\frac{\xi}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)}+\frac{\xi t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right]\\ &+(1-\delta_{ij})\frac{\left[-\frac{\partial}{\partial\phi}\frac{\partial}{\partial\beta_1}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right)-\frac{\partial}{\partial\beta_1}F_{\xi_i}(\eta)\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\end{aligned} \quad (8)$$

Similar as in Cox and Hinkley (1973), section 4.8 we have

$E\left[\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_0}\right]$=0, then

$$\begin{aligned}&\frac{1}{2}F_{\xi_i}(\eta)-\frac{1}{2}\phi E\left(\frac{\delta_{ij}}{t_{ij}}\right)e^{(\beta_0+\beta_1\xi_i)}+\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}\phi E\left(\delta_{ij}t_{ij}\right)-\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)=0\\ \Rightarrow\quad &-\frac{1}{2}\phi E\left(\frac{\delta_{ij}}{t_{ij}}\right)e^{(\beta_0+\beta_1\xi_i)}+\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}\phi E\left(\delta_{ij}t_{ij}\right)=\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)-\frac{1}{2}F_{\xi_i}(\eta)\end{aligned} \quad (9)$$

$E\left[\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \beta_1}\right]$=0, then

$$\begin{aligned}&\frac{1}{2}\xi_iF_{\xi_i}(\eta)-\frac{1}{2}\xi_i\phi E\left(\frac{\delta_{ij}}{t_{ij}}\right)e^{(\beta_0+\beta_1\xi_i)}+\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}\xi_i\phi E\left(\delta_{ij}t_{ij}\right)-\frac{\partial}{\partial\beta_1}F_{\xi_i}(\eta)=0\\ \Rightarrow\quad &-\frac{1}{2}\xi_i\phi E\left(\frac{\delta_{ij}}{t_{ij}}\right)e^{(\beta_0+\beta_1\xi_i)}+\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}\xi_i\phi E\left(\delta_{ij}t_{ij}\right)=\frac{\partial}{\partial\beta_1}F_{\xi_i}(\eta)-\frac{1}{2}\xi_iF_{\xi_i}(\eta)\end{aligned} \quad (10)$$

$E\left[\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial \phi}\right]$=0, then

$$\begin{aligned}&\frac{1}{2\phi}F_{\xi_i}(\eta)-\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}E\left(\delta_{ij}t_{ij}\right)+F_{\xi_i}(\eta)-\frac{e^{(\beta_0+\beta_1\xi_i)}}{2}E\left(\frac{\delta_{ij}}{t_{ij}}\right)-\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)=0\\ \Rightarrow\quad &\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}E\left(\delta_{ij}t_{ij}\right)+\frac{e^{(\beta_0+\beta_1\xi_i)}}{2}E\left(\frac{\delta_{ij}}{t_{ij}}\right)=\frac{1}{2\phi}F_{\xi_i}(\eta)+F_{\xi_i}(\eta)-\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\end{aligned} \quad (11)$$

We take an expectation of negative equation 3:

$$\begin{aligned}
E\left[-\frac{\partial}{\partial\beta_0}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial\beta_0}\right] =& E\left\{-\delta_{ij}\left[-\frac{\phi}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)}-\frac{\phi t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right]\right\} \\
&+E\left\{-\left(1-\delta_{ij}\right)\frac{\left[-\frac{\partial^2}{\partial\beta_0^2}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right)-\left(\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\right)^2\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\right\} \\
=&\frac{1}{2}\phi E\left(\frac{\delta_{ij}}{t_{ij}}\right)e^{(\beta_0+\beta_1\xi_i)}+\frac{\phi}{2e^{(\beta_0+\beta_1\xi_i)}}E\left(\delta_{ij}t_{ij}\right)+\frac{\partial^2F_{\xi_i}(\eta)}{\partial\beta_0^2} \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\left[\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\right]^2
\end{aligned}$$

and use equation 11 to replace unknown $E\left(\frac{\delta_{ij}}{t_{ij}}\right)$and $E\left(\delta_{ij}t_{ij}\right)$, then we obtain

$$\begin{aligned}
E\left[-\frac{\partial}{\partial\beta_0}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial\beta_0}\right] =& \phi\left(\frac{1}{2\phi}F_{\xi_i}(\eta)+F_{\xi_i}(\eta)-\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\right)+\left[1-F_{\xi_i}(\eta)\right]^{-1}\left[\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\right]^2 \\
=&\frac{1}{2}F_{\xi_i}(\eta)+\phi F_{\xi_i}(\eta)-\phi\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)+\left[1-F_{\xi_i}(\eta)\right]^{-1}\left[\frac{\partial F_{\xi_i}(\eta)}{\partial\beta_0}\right]^2
\end{aligned}$$

We take an expectation of negative equation 7:

$$\begin{aligned}
E\left[-\frac{\partial}{\partial\phi}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial\beta_0}\right] =& E\left\{-\delta_{ij}\left[-\frac{1}{2t_{ij}}e^{(\beta_0+\beta_1\xi_i)}+\frac{t_{ij}}{2e^{(\beta_0+\beta_1\xi_i)}}\right]\right\} \\
&+E\left\{-\left(1-\delta_{ij}\right)\frac{\left[-\frac{\partial}{\partial\phi}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\left(1-F_{\xi_i}(\eta)\right)-\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)\right]}{\left(1-F_{\xi_i}(\eta)\right)^2}\right\} \\
=&\frac{1}{2}E\left(\frac{\delta_{ij}}{t_{ij}}\right)e^{(\beta_0+\beta_1\xi_i)}-\frac{1}{2e^{(\beta_0+\beta_1\xi_i)}}E\left(\delta_{ij}t_{ij}\right)+\frac{\partial}{\partial\phi}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta) \\
&+\left[1-F_{\xi_i}(\eta)\right]^{-1}\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)
\end{aligned}$$

and use equation 9 to replace unknown $E\left(\frac{\delta_{ij}}{t_{ij}}\right)$and $E\left(\delta_{ij}t_{ij}\right)$, then we obtain

$$\begin{aligned}
E\left[-\frac{\partial}{\partial\phi}\frac{\partial L_{ij}(\boldsymbol{\theta})}{\partial\beta_0}\right] =& -\frac{1}{\phi}\left(\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)-\frac{1}{2}F_{\xi_i}(\eta)\right)+\frac{\partial}{\partial\phi}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta) \\
&+\frac{1}{2}\left[1-F_{\xi_i}(\eta)\right]^{-1}\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta) \\
=& -\frac{1}{\phi}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)+\frac{1}{2\phi}F_{\xi_i}(\eta)+\frac{\partial}{\partial\phi}\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta) \\
&+\frac{1}{2}\left[1-F_{\xi_i}(\eta)\right]^{-1}\frac{\partial}{\partial\phi}F_{\xi_i}(\eta)\frac{\partial}{\partial\beta_0}F_{\xi_i}(\eta)
\end{aligned}$$

By the same way, we derived the Fisher information matrix $I_{ij}(\boldsymbol{\theta})$.

3.2 Delta Method

Meeker and Escobar (1998, Appendix) consider $g(\boldsymbol{\theta})$ be a real-valued function of parameter $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_n)'$. The parameters can be estimated to find $\widehat{\boldsymbol{\theta}}$ and $g(\widehat{\boldsymbol{\theta}})$, respectively. When $g(\boldsymbol{\theta})$ has a continuous second partial derivative in respect of $\boldsymbol{\theta}$, the delta method can provide:

$$\begin{aligned} E[g(\widehat{\boldsymbol{\theta}})] &\approx g(\boldsymbol{\mu}) \\ Var[g(\widehat{\boldsymbol{\theta}})] &\approx [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}]' \ \Sigma_{\widehat{\boldsymbol{\theta}}} \ [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}] \\ or \ \ Var[g(\widehat{\boldsymbol{\theta}})] &\approx [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}]' \ I_{\boldsymbol{\theta}}^{-1} \ [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}] \end{aligned}$$

where $\boldsymbol{\mu} = [E(\widehat{\theta_1}), E(\widehat{\theta_1}), ..., E(\widehat{\theta_n})]$ and $\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_1}, ..., \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_n}] \neq 0$. The $I_{\boldsymbol{\theta}}$ given above is often known as the Fisher information. Asymptotic (large-sample) theory shows that, under the standard regularity conditions:

$$\begin{aligned} \sqrt{n}(g(\widehat{\boldsymbol{\theta}}) - g(\boldsymbol{\theta})) &\xrightarrow{D} \mathbf{N}(0, [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}]' \ n\Sigma_{\widehat{\boldsymbol{\theta}}} \ [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}]) \\ or \ \ \sqrt{n}(g(\widehat{\boldsymbol{\theta}}) - g(\boldsymbol{\theta})) &\xrightarrow{D} \mathbf{N}(0, [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}]' \ nI_{\boldsymbol{\theta}}^{-1} \ [\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}]) \end{aligned}$$

When β_0, β_1, ϕ be calculated to be $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ $(MLEs)$, then $\widehat{F_\xi}(t; \widehat{\boldsymbol{\theta}})$ can be expressed as below:

$$\widehat{F_\xi}(t; \widehat{\boldsymbol{\theta}}) = \Phi \left[\sqrt{\frac{e^{\left(\widehat{\beta_0} + \widehat{\beta_1}\xi\right)} \widehat{\phi}}{t}} \left(\frac{t}{e^{\left(\widehat{\beta_0} + \widehat{\beta_1}\xi\right)}} - 1 \right) \right] + e^{2\widehat{\phi}} \Phi \left[-\sqrt{\frac{e^{\left(\widehat{\beta_0} + \widehat{\beta_1}\xi\right)} \widehat{\phi}}{t}} \left(\frac{t}{e^{\left(\widehat{\beta_0} + \widehat{\beta_1}\xi\right)}} + 1 \right) \right]$$

and the asymptotic distribution of $\widehat{t_p}$ can be presented as:

$$\widehat{t_p} \sim \mathcal{N} \left[t_p, \frac{1}{\left[f_\xi \left(F_\xi^{-1}(p) \right) \right]^2} \left(\nabla F_\xi(t_p) \right)' \left(I(\boldsymbol{\theta}) \right)^{-1} \left(\nabla F_\xi(t_p) \right) \right]$$

where

$$\nabla F_\xi(t_p) = \left[\frac{\partial F_\xi(t_p)}{\partial \beta_0}, \frac{\partial F_\xi(t_p)}{\partial \beta_1}, \frac{\partial F_\xi(t_p)}{\partial \phi} \right]$$

4 Accurate Standard Deviation

Meeker(1984) suggested 1000 units tested during the experiment, then asymptotic normal theory can be applied. He used method in section 3 to determine the optimum plan. However when sample size is smaller (say less than 1000), the asymptotic variance derived by Fisher information may not be accurate enough for the standard deviation of $\widehat{t_p}$. The optimum plan founded may not really optimum in the sense. We reproduce the similar result given by Meeker (1984) but for different distribution and add the standard deviation (SD) obtained by the optimum plans. The censored time is fixed at $\eta = 4$, under several different sets of parameters, the highest stress censored proportion ranges from 0.272 to 1.0. See table 10 and table 11 for results of different parameters (β_0, β_1, ϕ, select by ourself), $t_{.1}$ means the quantile at 0.1. Subscript L, M and H denote the lowest, median and highest levels. The ξ and π denote stress and allocation, respectively. The p expresses the failure probability, $E(r)$ is the expected failure number. The p_0 is the failure probability at the design level (normal level). SD mean the standard deviation of $t_{.1}$. $R_{SD} = SD_{BCP}/SD_{OP}$, where BCP is Best Compromised Plan and OP is Optimum Plan. In the design stage, we consider some possible parameters in the model. See table 10, these parameters cover the possible range of usage. Under these parameter, we can calculate accurate standard deviation of the estimator by simulation. The accurate standard deviation can lead to the optimum plan we request. When sample size is not large enough (such as n=500 or 200), AN method and simulation method provide different optimum plans. Simulation gives more reliable result in small or moderate sample cases.

Table 1: The Optimum Plans for estimating $t_{.1}$ when sample size n=1000.

β_0	β_1	ϕ	$t_{.1}$	ξ_L	π_L	p_L	$E(r_L)$	p_H	$E(r_H)$	p_0	$SD_{t_{.1}}$
4	-2	1	13.0	0.363	0.70	0.026	0.018	0.40	0.120	$5.80e^{-004}$	1.503
4	-6	1	13.0	0.251	0.825	0.193	0.163	1.0	0.175	$5.80e^{-004}$	0.684
6	-4	1.5	123	0.739	0.607	0.020	0.012	0.326	0.128	$4.04e^{-034}$	39.79
6	-8	1.5	123	0.483	0.706	0.261	0.184	1.0	0.294	$4.04e^{-034}$	9.284

Table 2: The Best Compromised Plans for estimating $t_{.1}$ when sample size n=1000.

β_0	β_1	ϕ	$t_{.1}$	ξ_L	π_L	p_L	$E(r_L)$	p_M	$E(r_M)$	p_H	$E(r_H)$	$SD_{t_{.1}}$
4	-2	1	13.0	0.286	0.333	0.014	0.005	0.129	0.043	0.40	0.133	1.708
4	-6	1	13.0	0.231	0.333	0.159	0.053	0.951	0.317	1.0	0.333	0.906
6	-4	1.5	123	0.702	0.333	0.010	0.003	0.094	0.031	0.326	0.109	44.22
6	-8	1.5	123	0.463	0.333	0.191	0.064	0.980	0.327	1.0	0.333	11.53

We have same value p_0 as Optimum Plans.

5 Simulation Procedure

5.1 Generate the IG Random Number

If T is a random variable follow distribution F_T with parameter θ, then the relationship between cdf $F_T(t, \theta)$ and quantile is $F_T(t_p, \theta) = p$. Take the inverse function we have $t_p = F^{-1}(p, \theta)$. Mostly one obtains p from uniform distribution, this is the easiest way to generate random variables for quantiles. There is no close form of the IG inverse CDF. We apply the method in the early period, however, it turns out that to solve the inverse function is quite time consuming when the simulation number is large, say $n_s = 6000$. We turn to the method provided by Michael et al. (1976) to generate the IG random variable. The method is to find a transformation of the random variable, then apply the binomial distribution to choose one root for the interested value. Here is the procedure:

Let $T \sim IG(\mu, \phi)$, the transformed variable

$$Y^2 = \frac{\phi(T-\mu)^2}{\mu T}$$

distributed as χ^2_1 has two roots, T_1 and T_2, respectively, where

$$T_1 = \frac{1}{2\phi}\left[2\mu\phi + \mu Y^2 - \sqrt{4\mu^2\phi Y^2 + \mu^2 Y^4}\right]$$

and

$$T_2 = \frac{\mu^2}{T_1}$$

The whole procedure for generating IG variates is as the given tracks below:

1. Generate random numbers from the Chi-square distribution with 1 degree of freedom.

2. From step 1, compute the smallest root T_1 from above equation for each random value.

3. Take a bernoulli trial with probability of "success" $p = \mu / (\mu + T_1)$.

4. Eventually, the smallest root T_1 should be chosen while the trial results in a success, otherwise the large root T_2 is chosen.

By using this method to generate the IG random numbers, we cut the simulation time down to one-third of the time by solving the inverse IG CDF. It takes about 1 second to generate 1000 IG random numbers in the S+ software.

5.2 Number of Simulation

Suppose a random variable T follows the IG distribution with the parameter $\boldsymbol{\theta}$ (a vector). In the design stage, one would use possible sets of parameters to evaluate standard deviation of estimators. We select 4 sets of parameters to explore the difference of optimum design by using simulation and AN theory. Table 1 shows the 4 sets of parameters we consider and the results of Optimum Design based on AN theory. Table 2 shows the simulation content for the Best Compromised Design. In order to know the number of simulation (n_s) for needed precision, we compare the standard deviation of $\widehat{t_{.1}}$ for the Optimum Design n_s=6000, 10000 and 15000 and sample size n=1000, 500, 200. See tables 3, 4, 5. The last column SD_{SD} means the standard deviation of $SD_{\widehat{t_{.1}}}$. The standard deviation is at least accurate to the third digit for most sets of parameters except the $(6, -4, 1.5)$. Still it is accurate up to the second digit. We also do the comparison for the Best Compromised Design and we have the similar result. See tables 6, 7, 8.

5.3 Simulation Result

Table 12 shows the standard deviation of $\widehat{t_{.1}}$ from the 2 stress plan on the parameters $\beta_0 = 4$, $\beta_1 = -2$, $\phi = 1$, sample size n=1000, the number of simulation n_s=6000 and $\xi = 0.352$, $\pi = 0.829$ (the values obtained from section 4 based on AN theory), censored time $\eta = 4$. Tables 13, 14 are two similar tables with different sample size

$n = 500$ and $n = 200$. The horizontal axis denotes lower stress ξ_L and the vertical axis denotes lower allocation π_L. The number with a $\star$ is the standard deviation given the stress level ξ_L and allocation π_L obtained from the Optimum Plan by asymptotic normal theory. The $\xi_L^{'}$ and $\pi_L^{'}$ are the chosen values for the center of the table. The number with a $\star\star$ is the smallest standard deviation by a grid search for different lower stress level ξ_L and lower allocation π_L using simulation. We can see that Optimum Design should be a plan with lower stress $\xi_L^{''}$=ξ_L=0.363 and lower allocation $\pi_L^{''}$= π_L=0.70.

This provide an important information that the Optimum Plan provided by using AN theory is not truly optimum. By simulation we find the test plan more closer to the required optimum plan. When sample size is less than 1000, we also obtain the similar results, see table 13, 14. For different sets of parameters, the situation are the same. See tables 15 to 23.

For 3 stress Compromised Plan which gives equal amount of allocation for each stress, the only variable to decide the Best Compromised plan is the lower stress ξ_L. Table 24 shows the standard deviation of $\widehat{t_{.1}}$ from the 3 stress Compromised plan on the parameters $\beta_0 = 4$, $\beta_1 = -2$, $\phi = 1$, sample sizes include n=1000, 500 and 200, the number of simulation n_s=6000 and $\xi = 0.2865$, π at $0.33\bar{3}$ (the value obtained from section 4), censored time $\eta = 4$. The vertical axis denotes lower stress ξ_L. The number with a $\star$ is the standard deviation given the lowest stress level ξ_L obtained from the Best Compromise Plan by asymptotic normal theory. The number with a $\star\star$ is the smallest standard deviation by a grid search for different lower stress level ξ_L using simulation. We can see that Best design should be a plan with lower stress $\xi_L^{''}$=$\xi_L^{'}$+0.04= 0.291+0.04=0.295 for sample size n=1000.

This provide an important information that the Best Compromised Plan provided by using AN theory is not truly optimum. By simulation we find the test plan more closer to the required Optimum Plan. When sample size is less than 1000, we also obtain the similar results. For different set of parameters, the situation are the same.

Table 3: The table shows the standard deviation of $\widehat{t_{.1}}$ at the Two Stresses Plan for sample size of the designs $n = 1000$ and the number of simulation n_s= 6000, 10000, 15000 respectively. The parameters are (4,-2,1.0); (4,-6,1.0); (6, -4, 1.5); (6,-8,1.5)

(β_0, β_1, ϕ)	n_s=6000	n_s=10000	n_s=15000	SD_{SD}
(4,-2,1.0)	1.651	1.661	1.673	0.019
(4,-6,1.0)	0.705	0.701	0.706	0.006
(6,-4,1.5)	47.73	48.49	48.91	0.940
(6,-8,1.5)	9.446	9.467	9.540	0.102

See table 25 until 27.

6 Bias Correction by Bootstrap

6.1 Bootstrap procedure

The following sections use the “bootstrap principle” or Monte Carlo to find bias of parameter and correct bias. Suppose a statistic $F(\boldsymbol{\theta})$ has a distribution with parameter $\boldsymbol{\theta}$. The parametric bootstrap version $F^*(\widehat{\boldsymbol{\theta}})$ of F is the same function but evaluated at data (“bootstrap samples”) simulated using an estimate $\widehat{\boldsymbol{\theta}}$ in place of the unknown $\boldsymbol{\theta}$ [see Sec. 6.5, Efron and Tibshirani (1993) for more details]. Using $\widehat{\boldsymbol{\theta}}$ in place of the distribution parameters, the distribution of F^* can be calculated analytically in simple situations, or by simulation in general. We consider the scheme in section 5, and provide result of bias correction by bootstrap below.

6.2 Bias Correction

Suppose the distribution we are interested in is $F(t; \boldsymbol{\theta})$, $\boldsymbol{\theta}$ be the unknown parameter. Let $\widehat{\boldsymbol{\theta}}$ be the estimator of $\boldsymbol{\theta}$, the term $E(\widehat{\boldsymbol{\theta}}) - \boldsymbol{\theta}$ is called bias. The bias corrected

Table 4: The table shows the standard deviation of $\widehat{t_{.1}}$ at the Two Stresses Plan for sample size of the designs n = 500 and the number of simulation n_s= 6000, 10000, 15000 respectively. The parameters are (4,-2,1.0); (4,-6,1.0); (6, -4, 1.5); (6,-8,1.5)

(β_0, β_1, ϕ)	n_s=6000	n_s=10000	n_s=15000	SD_{SD}
(4,-2,1.0)	2.516	2.467	2.479	1.003
(4,-6,1.0)	1.015	1.004	1.003	0.011
(6,-4,1.5)	$5.968e^3$	$4.968e^3$	$4.057e^3$	1304
(6,-8,1.5)	13.65	13.70	13.61	0.153

Table 5: The table shows the standard deviation of $\widehat{t_{.1}}$ at the Two Stresses Plan for sample size of the designs n = 200 and the number of simulation n_s= 6000, 10000, 15000 respectively. The parameters are (4,-2,1.0); (4,-6,1.0); (6, -4, 1.5); (6,-8,1.5)

(β_0, β_1, ϕ)	n_s=6000	n_s=10000	n_s=15000	SD_{SD}
(4,-2,1.0)	38.36	36.08	34.70	2.122
(4,-6,1.0)	1.60	1.611	1.611	0.014
(6,-4,1.5)	$4.588e^4$	4.440^4	$4.246e^4$	1912
(6,-8,1.5)	21.50	22.02	22.0	0.261

Table 6: The table shows the standard deviation of $\widehat{t_{.1}}$ at the Three Stresses Plan for sample size of the designs n = 1000 and the number of simulation n_s= 6000, 10000, 15000 respectively. The parameters are (4,-2,1.0); (4,-6,1.0); (6, -4, 1.5); (6,-8,1.5)

(β_0, β_1, ϕ)	n_s=6000	n_s=10000	n_s=15000	SD_{SD}
(4,-2,1.0)	1.815	1.843	1.866	0.025
(4,-6,1.0)	0.911	0.913	0.918	0.011
(6,-4,1.5)	54.27	55.99	56.64	1.533
(6,-8,1.5)	11.65	11.64	11.67	0.138

Table 7: The table shows the standard deviation of $\widehat{t_{.1}}$ at the Three Stresses Plan for sample size of the designs n = 500 and the number of simulation n_s= 6000, 10000, 15000 respectively. The parameters are (4,-2,1.0); (4,-6,1.0); (6, -4, 1.5); (6,-8,1.5)

(β_0, β_1, ϕ)	n_s=6000	n_s=10000	n_s=15000	SD_{SD}
(4,-2,1.0)	2.858	2.830	2.794	0.035
(4,-6,1.0)	1.323	1.309	1.298	0.017
(6,-4,1.5)	103.6	101.5	103.3	5.728
(6,-8,1.5)	16.73	16.59	16.57	0.215

Table 8: The table shows the standard deviation of $\widehat{t_{.1}}$ at the Three Stresses Plan for sample size of the designs n = 200 and the number of simulation n_s= 6000, 10000, 15000 respectively. The parameters are (4,-2,1.0); (4,-6,1.0); (6, -4, 1.5); (6,-8,1.5)

(β_0, β_1, ϕ)	n_s=6000	n_s=10000	n_s=15000	SD_{SD}
(4,-2,1.0)	6.604	6.536	6.284	103.3
(4,-6,1.0)	2.10	2.125	2.141	0.029
(6,-4,1.5)	$2.861e^6$	$2.835e^6$	$2.663e^6$	883851
(6,-8,1.5)	27.30	27.65	27.63	0.396

Table 9: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Two stresses Plan on the parameters (β_0, β_1, ϕ)= (4,-2,1);(4,-6,1);(6,-4,1.5);(6,-8,1.5), sample size n=1000; 500; 200, the number of simulation n_s=6000

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	(4,-2,1)	(4,-6,1)	(6,-4,1.5)	(6,-8,1.5)
n=1000	(0.35,0.83,1.651)	(0.22,0.85,0.705)	(0.74,0.70,47.73)	(0.46,0.73,9.446)
	(0.39,0.75,1.568)	(0.26,0.81,0.689)	(0.78,0.66,47.69)	(0.50,0.69,9.303)
n=500	(0.35,0.83,2.516)	(0.22,0.85,1.015)	(0.74,0.70,5968)	(0.46,0.73,13.65)
	(0.39,0.70,2.381)	(0.26,0.81,0.976)	(0.78,0.74,84.20)	(0.50,0.69,13.46)
n=200	(0.35,0.83,38.36)	(0.22,0.85,1.60)	(0.74,0.69,$4.583e^4$)	(0.46,0.73,21.50)
	(0.47,0.83,4.350)	(0.26,0.85,1.556)	(0.86,0.81,309.4)	(0.50,0.69,21.18)

estimator can be $\widehat{\boldsymbol{\theta}} - \left(E(\widehat{\boldsymbol{\theta}}) - \widehat{\boldsymbol{\theta}}\right)$. For estimating $t_{.1}$, we can use $2\widehat{t_{.1}} - E\left(\widehat{t_{.1}}\right)$ as its bias corrected estimator. Under time censoring it is difficult to calculate $E\left(\widehat{t_{.1}}\right)$. We can apply bootstrap procedure to obtain an estimate of $E\left(\widehat{t_{.1}}\right)$, say $\overline{\widehat{t^*_{.1}}}$. Then $\widehat{b}$=$\overline{\widehat{t^*_{.1}}} - \widehat{t_{.1}}$ is a bias estimate.

6.3 Results

To compare the true bias, we calculate $E\left(\widehat{t_{.1}}\right) - t_{.1}$ by simulation $n_s = 6000$ to give $\overline{\widehat{t_{.1}}} - t_{.1}$. See tables 28, 29, 30, 31, there are 4 sets of parameters for sample size $n = 1000$, 500 and 200. Tables 36, 37, 38, 39 are bias correction results for Optimum Plans from simulation. For each set of parameters we consider the bias for sample size $n = 1000$, 500 and 200. For each sample size 3 simulations were given to see the standard deviation of the ML estimate. We see that the estimate bias is positive for almost all cases.

Comparing tables 28, 29, 30, 31, and table 36, 37, 38, 39, we notice that the bias correction is on the right direction. For the set of parameter (6,-4,1.5) it is clear that the correction is needed.

To compare the true bias, we calculate $E\left(\widehat{t_{.1}}\right) - t_{.1}$ by simulation $n_s = 6000$ to give $\overline{\widehat{t_{.1}}} - t_{.1}$. See tables 32, 33, 34, 35, there are 4 sets of parameters for sample size $n = 1000$, 500 and 200. Tables 40, 41, 42, 43 are bias correction results for Best Compromised Plans from simulation. For each set of parameters we consider the bias for sample size $n = 1000$, 500 and 200. For each sample size 3 simulations were given to see the standard deviation of the ML estimate. We see that the estimate bias is positive for almost all cases.

Comparing tables 32, 33, 34, 35, and tables 40, 41, 42, 43, we notice that the bias correction is on the right direction. For the set of parameter (6,-4,1.5) it is clear that the correction is needed.

7 Conclusion

In this paper we provide these conclusions.

1. In this paper we give the Fisher information matrix for IG distribution and prove the result in detail.

2. Optimum Plan and Best Compromised Plan cannot be provided by the method based on asymptotic normal theory. When sample size is not large enough, say ≤ 1000, we show that the required plans, can be obtained by simulation.

3. Bias correction has effect on the design for most parameters. However, for in some set of parameters, the bootstrap correction does not provide significant difference.

4. There are difficulties in the small sample for an accelerated life test plan, especially when the extrapolate location of the normal stress is far from the accelerated stress. The standard deviation of $\widehat{t_{.1}}$ is too large that the Optimum Plan and Best Compromised Plan are not proper for getting the estimate of product life.

8 Future Research

1. Compare with other possible distributions. For example Weibull and Lognormal.

2. Complete table for more possible parameters.

3. Consider the plans with smaller mean square error. This criterion can take care of the variance and bias of the estimator at the some time.

4. Different test plans, for example: Best Equal Expected Number Failing Plans.

5. Suppose the result of sample size too small, say less than 200. Develop the rules or criteria for obtaining the Optimum Plans.

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Table 10: The Optimum Plans for estimating $t_{.1}$ when sample size n=1000.

β_0	β_1	ϕ	$t_{.1}$	ξ_L	π_L	p_L	$E(r_L)$	p_H	$E(r_H)$	p_0	$SD_{t_{.1}}$	R_{SD}
4	-2	1	13.0	0.363	0.70	0.026	0.018	0.40	0.120	$6e^{-004}$	1.503	1.137
4	-4	1	13.0	0.303	0.782	0.109	0.085	0.979	0.213	$6e^{-004}$	0.869	1.251
4	-6	1	13.0	0.251	0.825	0.197	0.163	1.0	0.175	$6e^{-004}$	0.684	1.325
6	-4	1	95.9	0.674	0.627	0.023	0.015	0.40	0.149	$3e^{-023}$	24.93	1.129
6	-6	1	95.9	0.534	0.677	0.106	0.072	0.979	0.316	$3e^{-023}$	11.14	1.189
6	-8	1	95.9	0.443	0.719	0.212	0.153	1.0	0.281	$3e^{-023}$	7.717	1.241
8	-6	1	708	0.782	0.604	0.023	0.014	0.40	0.158	$1e^{-163}$	286.9	1.130
8	-8	1	708	0.650	0.634	0.106	0.067	0.979	0.358	$1e^{-163}$	117.1	1.178
8	-10	1	708	0.556	0.668	0.218	0.146	1.0	0.332	$1e^{-163}$	76.77	1.211
4	-2	1.5	16.7	0.487	0.655	0.022	0.014	0.326	0.112	$3e^{-005}$	2.487	1.109
4	-4	1.5	16.7	0.384	0.748	0.132	0.099	0.989	0.249	$3e^{-005}$	1.140	1.223
4	-6	1.5	16.7	0.301	0.801	0.235	0.189	1.0	0.199	$3e^{-005}$	0.833	1.312
6	-4	1.5	123	0.739	0.607	0.020	0.012	0.326	0.128	$4e^{-034}$	39.79	1.111
6	-6	1.5	123	0.589	0.658	0.132	0.087	0.989	0.338	$4e^{-034}$	14.36	1.175
6	-8	1.5	123	0.483	0.706	0.261	0.184	1.0	0.294	$4e^{-034}$	9.284	1.242
8	-6	1.5	910	0.841	0.602	0.029	0.017	0.326	0.130	$2e^{-244}$	457.3	1.103
8	-8	1.5	910	0.692	0.621	0.133	0.082	0.989	0.375	$2e^{-244}$	149.9	1.163
8	-10	1.5	910	0.589	0.660	0.273	0.180	1.0	0.340	$2e^{-244}$	91.84	1.215
4	-2	2	19.5	0.567	0.630	0.019	0.012	0.272	0.101	$1e^{-006}$	3.50	1.096
4	-4	2	19.5	0.435	0.729	0.153	0.112	0.994	0.270	$1e^{-006}$	1.319	1.212
4	-6	2	19.5	0.330	0.787	0.263	0.207	1.0	0.213	$1e^{-006}$	0.916	1.308
6	-4	2	144	0.780	0.596	0.017	0.010	0.272	0.110	$6e^{-045}$	54.82	1.101
6	-6	2	144	0.625	0.648	0.156	0.101	0.994	0.350	$6e^{-045}$	16.45	1.171
6	-8	2	144	0.505	0.699	0.295	0.206	1.0	0.301	$6e^{-045}$	10.13	1.246
8	-6	2	$1.1e^3$	0.853	0.584	0.017	0.010	0.272	0.113	$0e^{-000}$	619.6	1.104
8	-8	2	$1.1e^3$	0.719	0.614	0.157	0.097	0.994	0.384	$0e^{-000}$	171.0	1.172
8	-10	2	$1.1e^3$	0.607	0.655	0.311	0.204	1.0	0.345	$0e^{-000}$	99.89	1.222

Table 11: The Best Compromised Plans for estimating $t_{.1}$ when sample size n=1000.

β_0	β_1	ϕ	$t_{.1}$	ξ_L	π_L	p_L	$E(r_L)$	p_M	$E(r_M)$	p_H	$E(r_H)$	$SD_{t_{.1}}$
4	-2	1	13.0	0.286	0.333	0.014	0.005	0.129	0.043	0.40	0.133	1.708
4	-4	1	12.0	0.270	0.333	0.078	0.026	0.638	0.213	0.979	0.326	1.088
4	-6	1	12.0	0.231	0.333	0.159	0.053	0.951	0.317	1.0	0.333	0.906
6	-4	1	95.9	0.637	0.333	0.013	0.005	0.125	0.042	0.40	0.133	28.14
6	-6	1	95.9	0.510	0.333	0.075	0.025	0.634	0.211	0.979	0.326	13.24
6	-8	1	95.9	0.422	0.333	0.155	0.052	0.950	0.317	1.0	0.333	9.580
8	-6	1	708	0.757	0.333	0.013	0.004	0.124	0.041	0.40	0.133	324.1
8	-8	1	708	0.647	0.333	0.10	0.033	0.658	0.212	0.980	0.326	138.0
8	-10	1	708	0.537	0.333	0.153	0.051	0.950	0.317	1.0	0.333	92.97
4	-2	1.5	16.7	0.412	0.333	0.011	0.004	0.096	0.032	0.326	0.109	2.758
4	-4	1.5	16.7	0.344	0.333	0.087	0.029	0.679	0.226	0.989	0.330	1.394
4	-6	1.5	16.7	0.284	0.333	0.183	0.061	0.980	0.327	1.0	0.333	1.093
6	-4	1.5	123	0.702	0.333	0.010	0.003	0.094	0.031	0.326	0.109	44.22
6	-6	1.5	123	0.561	0.333	0.084	0.028	0.677	0.226	0.989	0.330	16.87
6	-8	1.5	123	0.463	0.333	0.191	0.064	0.980	0.327	1.0	0.333	11.53
8	-6	1.5	910	0.80	0.333	0.010	0.003	0.093	0.031	0.326	0.109	504.2
8	-8	1.5	910	0.670	0.333	0.084	0.028	0.676	0.225	0.989	0.330	174.3
8	-10	1.5	910	0.570	0.333	0.191	0.064	0.980	0.327	1.0	0.333	111.6
4	-2	2	19.5	0.495	0.333	0.009	0.003	0.076	0.025	0.272	0.091	3.835
4	-4	2	19.5	0.393	0.333	0.096	0.032	0.720	0.240	0.994	0.331	1.598
4	-6	2	19.5	0.317	0.333	0.224	0.075	0.993	0.331	1.0	0.333	1.199
6	-4	2	144	0.744	0.333	0.008	0.003	0.074	0.025	0.272	0.091	60.36
6	-6	2	144	0.594	0.333	0.095	0.032	0.718	0.240	0.994	0.331	19.26
6	-8	2	144	0.488	0.333	0.226	0.075	0.993	0.331	1.0	0.333	12.63
8	-6	2	$1.1e^3$	0.829	0.333	0.008	0.003	0.074	0.025	0.272	0.091	683.7
8	-8	2	$1.1e^3$	0.717	0.333	0.152	0.051	0.760	0.253	0.994	0.331	200.4
8	-10	2	$1.1e^3$	0.594	0.333	0.242	0.081	0.993	0.331	1.0	0.333	122.1

We have same value p_0 as Optimum Plans.

Table 12: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=4, β_1=−2, ϕ=1 by simulation, sample size n=1000, the number of simulation n_s=6000 and ξ_L=0.363, π_L=0.70, $\xi_L^{'}$=0.352, $\pi_L^{'}$=0.829, SD=1.503

	-.12	-.08	-.04	$\xi_L^{'}$	+.01	+.04	+.08	+.12	+.16
+.12	2.272	2.224	2.189	2.191	*	2.216	2.265	2.336	*
+.08	2.330	1.919	1.904	1.892	*	1.926	1.914	1.964	*
+.04	3.487	2.444	1.760	1.743	*	1.744	1.780	1.814	*
$\pi_L^{'}$	3.818	2.399	1.684	1.651	*	1.646	1.684	1.717	*
-.04	4.188	2.333	1.641	1.603	*	1.592	1.623	1.648	1.723
-.07	*	*	*	*	1.559$^{\star\star}_{\star}$	*	*	*	*
-.08	4.354	2.250	1.629	1.582	*	1.568	1.592	1.610	1.676
-.12	4.705	2.256	1.641	1.582	*	1.570	1.586	1.609	1.667
-.16	*	*	1.649	1.593	*	1.573	1.589	1.613	*
-.20	*	*	1.679	1.607	*	1.594	1.614	1.637	*

Table 13: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=4, β_1=−2, ϕ=1 by simulation, sample size n=500, the number of simulation n_s=6000 and ξ_L=0.363, π_L=0.70, $\xi_L^{'}$=0.352, $\pi_L^{'}$=0.829, SD=2.215

	-.12	-.08	-.04	$\xi_L^{'}$	+.01	+.04	+.08	+.12	+.16
+.12	14.58	8.109	5.690	3.147	*	3.147	3.243	3.372	*
+.08	15.46	7.710	5.228	2.757	*	2.735	2.795	2.871	*
+.04	15.43	8.352	5.052	2.597	*	2.552	2.692	2.681	*
$\pi_L^{'}$	16.60	9.016	5.101	2.516	*	2.468	2.512	2.564	*
-.04	17.76	10.33	5.201	2.454	*	2.411	2.434	2.479	2.559
-.07	*	*	*	*	4.627⋆	*	*	*	*
-.08	19.30	11.63	7.583	4.225	*	2.381⋆⋆	2.390	2.419	2.496
-.12	21.05	13.19	9.103	5.402	*	2.389	2.388	2.396	2.462
-.16	*	*	*	5.782	*	2.410	2.404	2.415	*
-.20	*	*	*	*	*	4.589	2.433	2.446	*

Table 14: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=4, β_1=−2, ϕ=1 by simulation, sample size n=200, the number of simulation n_s=6000 and ξ_L=0.363, π_L=0.70, $\xi_L^{'}$=0.352, $\pi_L^{'}$=0.829, SD=3.360

	-.12	-.08	-.04	$\xi_L^{'}$	+0.01	+.04	+.08	+.12	+.16	+.20	+.24
+.12	46.94	46.35	44.06	34.78	*	20.57	8.650	5.418	5.811	6.411	7.275
+.08	47.96	47.78	42.16	36.32	*	16.01	6.696	4.806	5.061	5.448	6.103
+.04	46.53	47.84	42.64	35.91	*	22.12	6.773	4.503	4.711	4.991	5.412
$\pi_L^{'}$	47.71	47.08	42.42	38.36	*	24.40	7.345	4.350⋆⋆	4.521	4.719	5.161
-.04	47.50	48.53	45.06	40.81	*	28.40	15.68	14.44	23.78	*	*
-.07	*	*	*	*	43.57⋆	*	*	*	*	*	*
-.08	48.94	50.40	48.42	45.06	*	33.59	19.68	13.73	*	*	*
-.12	48.19	50.83	50.73	45.77	*	34.43	24.30	16.98	*	*	*

Table 15: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation, sample size n=1000, the number of simulation n_s=6000 and ξ_L=0.251, π_L=0.825, $\xi_L^{'}$=0.215, $\pi_L^{'}$=0.854, SD=0.684

	-.12	-.08	-.04	$\xi_L^{'}$	+0.036	+.04	+.08	+.12	+.16
+.12	1.222	1.028	0.913	0.878	*	0.923	1.034	1.185	*
+.08	1.113	0.911	0.798	0.750	*	0.752	0.803	0.890	*
+.04	1.088	0.871	0.765	0.716	*	0.706	0.734	0.795	*
$\pi_L^{'}$	1.096	0.862	0.756	0.705	*	0.690	0.709	0.758	*
-.029	*	*	*	*	0.689*	*	*	*	*
-.04	1.117	0.862	0.756	0.705	*	0.689**	0.705	0.748	0.810
-.08	1.144	0.878	0.764	0.709	*	0.690	0.702	0.739	0.795
-.12	1.172	0.901	0.778	0.716	*	0.696	0.705	0.739	*
-.16	*	*	*	0.727	*	0.705	0.713	0.746	*
-.20	*	*	*	0.749	*	0.725	0.731	0.761	*

Table 16: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation, sample size n=500, the number of simulation n_s=6000 and ξ_L=0.251, π_L=0.825, $\xi_L^{'}$=0.215, $\pi_L^{'}$=0.854, SD=0.967

	-.12	-.08	-.04	$\xi_L^{'}$	+0.036	+.04	+.08	+.12	+.16
+.12	2.810	1.467	1.311	1.269	*	1.320	1.486	1.721	*
+.08	2.972	1.299	1.137	1.081	*	1.072	1.151	1.279	*
+.04	3.153	1.255	1.085	1.027	*	1.004	1.054	1.147	*
$\pi_L^{'}$	3.460	1.253	1.088	1.015	*	0.985	1.021	1.094	*
-0.029	*	*	*	*	0.983*	*	*	*	*
-.04	3.864	1.272	1.092	1.013	*	0.976**	1.005	1.065	1.154
-.08	4.486	1.299	1.108	1.028	*	0.991	1.013	1.064	1.144
-.12	5.435	1.319	1.124	1.036	*	1.0	1.020	1.068	1.142
-.16	*	*	*	1.062	*	1.022	1.037	1.080	*
-.20	*	*	*	1.082	*	1.042	1.058	1.093	*

Table 17: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation, sample size n=200, the number of simulation n_s=6000 and ξ_L=0.251, π_L=0.825, $\xi_L^{'}$=0.215, $\pi_L^{'}$=0.854, SD=1.529

	-.12	-.08	-.04	$\xi_L^{'}$	+0.036	+.04	+.08	+.12	+.16
+.12	*	*	1.978	1.932	*	2.055	2.325	2.674	*
+.08	20.86	4.291	1.799	1.678	*	1.679	1.794	1.992	*
+.04	22.37	4.552	1.767	1.612	*	1.582	1.647	1.790	*
$\pi_L^{'}$	23.46	5.796	1.778	1.60	*	1.556	1.602	1.718	1.874
-.029	*	*	*	*	$1.553^{\star}_{\star\star}$	*	*	*	*
-.04	24.71	6.443	1.798	1.611	*	1.558	1.591	1.687	1.820
-.08	25.37	7.436	1.819	1.626	*	1.569	1.590	1.671	*
-.12	26.43	8.943	1.859	1.648	*	1.587	1.596	1.662	*
-.16	*	*	*	1.696	*	1.624	1.621	*	*

Table 18: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=6, β_1=-4, ϕ=1.5 by simulation, sample size n=1000, the number of simulation n_s=6000 and ξ_L=0.739, π_L=0.607, $\xi_L^{'}$=0.741, $\pi_L^{'}$=0.697, SD=39.79

	-.12	-.08	-.04	-0.002	$\xi_L^{'}$	+.04	+.08	+.12
+.12	2054	1968	393.0	*	52.57	54.26	59.70	74.11
+.08	1933	1999	360.5	*	49.62	50.77	55.97	69.37
+.04	1954	2099	555.5	*	49.13	49.45	54.02	66.15
$\pi_L^{'}$	1912	2152	1052	*	47.73	47.74	52.01	63.16
-.04	1901	2247	1078	*	48.20	47.69**	51.46	61.97
-.08	1879	2374	1494	*	48.38	48.26	51.10	60.44
-.09	*	*	*	48.01*	*	*	*	*
-.12	1854	2516	1742	*	50.27	48.18	50.91	60.43
-.16	*	*	*	*	50.32	49.33	51.76	*

Table 19: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum plan on the parameters β_0=6, β_1=-4, ϕ=1.5 by simulation, sample size n=500, the number of simulation n_s=6000 and ξ_L=0.739, π_L=0.607, $\xi_L^{'}$=0.741, $\pi_L^{'}$=0.697, SD=56.28

	-.12	-.08	-.04	-.002	$\xi_L^{'}$	+.04	+.08	+.12
+.12	2948	5123	4553	*	1798	90.78	100.5	138.9
+.08	2553	4959	4965	*	1353	87.31	93.35	121.6
+.04	2433	4698	4952	*	6209	84.20**	89.72	114.1
$\pi_L^{'}$	2334	4675	5947	*	5968	84.94	90.10	112.3
-.04	2221	4575	6132	*	5445	85.63	88.46	106.4
-.08	2127	4525	6150	*	6647	85.28	86.11	106.8
-.09	*	*	*	6450*	*	*	*	*
-.12	2062	4580	6820	*	6953	85.63	87.98	110.6

Table 20: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum plan on the parameters β_0=6, β_1=−4, ϕ=1.5 by simulation, sample size n=200, the number of simulation n_s=6000 and ξ_L=0.739, π_L=0.607, $\xi^{'}$=0.741, $\pi^{'}$=0.697, SD=88.98

	-.12	-.08	-.04	-.002	$\xi_L^{'}$	+.04	+.08	+.12	+.16
+.20	*	*	*	*	*	$1.14e^4$	$4.50e^2$	$1.22e^3$	7.70^3
+.16	*	*	*	*	*	$1.45e^4$	$3.68e^2$	$7.89e^2$	$2.73e^3$
+.12	*	*	*	*	*	$3.60e^4$	$1.93e^4$	$3.09e^{2\star\star}$	$1.79e^3$
+.08	$4.99e^3$	$1.07e^4$	$2.22e^4$	*	$4.06e^4$	$3.01e^4$	$1.14e^4$	$4.52e^2$	$1.06e^3$
+.04	$4.48e^3$	$9.96e^3$	$2.13e^4$	*	$4.24e^4$	$3.22e^4$	$1.44e^4$	$3.78e^2$	$1.01e^3$
$\pi_L^{'}$	$3.02e^3$	$8.42e^3$	$2.06e^4$	*	$4.58e^{4\star}$	$4.02e^4$	$1.42e^4$	$3.96e^2$	$9.67e^2$
-.04	$3.07e^3$	7.75^3	2.0^4	*	$4.50e^4$	$4.98e^4$	$1.30e^4$	$4.62e^2$	*
-.08	$3.05e^3$	$7.38e^3$	$1.89e^4$	*	$4.50e^4$	$6.10e^4$	$2.47e^4$	$4.0e^2$	*
-.09	*	*	*	$4.5e^{4\star}$	*	*	*	*	*

Table 21: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum plan on the parameters β_0=6, β_1=−8, ϕ=1.5 by simulation, sample size n=1000, the number of simulation n_s=6000 and ξ_L=0.483, π_L=0.707, ξ_L'=0.455, π_L'=0.730, SD=9.284

	-.12	-.08	-.04	ξ_L'	+.028	+.04	+.08	+.12
+.12	115.1	13.39	11.0	10.05	*	10.07	10.97	12.32
+.08	131.2	13.12	10.70	9.746	*	9.723	10.49	11.70
+.04	143.0	13.04	10.54	9.558	*	9.504	10.19	11.27
π_L'	155.6	13.17	10.51	9.446	*	9.360	9.979	10.95
-.023	*	*	*	*	9.341⋆	*	*	*
-.04	165.6	13.30	10.53	9.414	*	9.303⋆⋆	9.874	10.77
-.08	182.7	13.70	10.72	9.583	*	9.403	9.926	10.79
-.12	191.6	14.05	10.90	9.660	*	9.457	9.938	10.78
-.16	*	*	*	9.870	*	9.647	10.11	10.92

Table 22: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=6, β_1=-8, ϕ=1.5 by simulation, sample size n=500, the number of simulation n_s=6000 and ξ_L=0.483, π_L=0.707, $\xi_L^{'}$=0.455, $\pi_L^{'}$=0.730, SD=13.13

	-.12	-.08	-.04	$\xi_L^{'}$	+0.028	+.04	+.08	+.12
+.12	336.6	20.11	16.03	14.47	*	14.50	15.71	17.67
+.08	349.0	19.92	15.57	13.96	*	13.90	14.90	16.61
+.04	360.2	20.05	15.42	13.81	*	13.64	14.450	16.03
$\pi_L^{'}$	370.8	20.15	15.26	13.65	*	13.48	14.18	15.65
-0.023	*	*	*	*	13.41⋆⋆	*	*	*
-.04	386.9	34.16	15.50	13.78	*	13.46	14.08	15.46
-.08	395.9	33.44	15.62	13.80	*	13.50	14.06	15.34
-.12	*	48.37	15.92	14.02	*	13.63	14.10	15.33

Table 23: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Optimum Plan on the parameters β_0=6, β_1=-8, ϕ=1.5 by simulation, sample size n=200, the number of simulation n_s=6000 and ξ_L=0.483, π_L=0.707, $\xi_L^{'}$=0.455, $\pi_L^{'}$=0.730, SD=20.76

	-.12	-.08	-.04	$\xi_L^{'}$	+0.028	+.04	+.08	+.12
+.12	557.5	283.2	25.85	22.59	*	22.58	24.31	27.41
+.08	550.9	316.1	25.32	21.92	*	21.82	23.20	25.90
+.04	546.4	345.9	25.10	21.62	*	21.44	22.59	25.13
$\pi_L^{'}$	541.8	366.6	25.31	21.50	*	21.25	22.27	24.48
-0.023	*	*	*	*	21.09⋆⋆	*	*	*
-.04	536.21	401.9	26.01	21.77	*	21.18	22.01	24.05
-.08	530.2	434.4	26.64	22.21	*	21.41	22.10	24.04
-.12	523.9	452.9	68.81	22.62	*	21.77	22.32	24.16

Table 24: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Best Compromise Plan on the parameters β_0=4, β_1=−2, ϕ=1, sample size n=1000, n=500, n=200, respectively. The number of simulation n_s=6000 and ξ_L=0.287, $\xi_L^{'}$=0.291, π=0.333, $SD_{1000} = 1.708$, $SD_{500} = 2.416$, $SD_{200} = 3.820$

	n=1000	n=500	n=200
+.28	*	*	6.106
+.24	*	*	5.791
+.20	1.972	3.145	5.612**
+.16	1.909	2.918	6.112
+.12	1.857	2.827	5.773
+.08	1.804	2.822**	5.814
+.04	1.795**	2.833	5.890
$\xi_L^{'}$	1.815	2.858	6.604
-.004	1.821*	2.853*	6.592*
-.04	1.821	2.871	6.631
-.08	1.866	2.935	153.8
-.12	1.897	2.975	119.3

Table 25: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Best Compromise Plan on the parameters β_0=4, β_1=-6, ϕ=1 by simulation ,sample size n=1000, n=500, n=200, respectively. The number of simulation n_s=6000 and ξ_L=0.2312, ξ_L'=0.232, π=0.333, $SD_{1000} = 0.906$, $SD_{500} = 1.281$, $SD_{200} = 2.026$

	n=1000	n=500	n=200
+.28	*	1.956	3.125
+.24	*	1.779	2.830
+.20	1.137	1.636	2.607
+.16	1.057	1.522	2.417
+.12	0.999	1.430	2.272
+.08	0.947	1.364	2.168
+.04	0.920	1.320**	2.110
ξ_L'	0.911	1.323	2.10**
-0.0008	0.910**	1.323*	2.103*
-.04	0.930	1.354	2.180
-.08	0.982	1.430	2.401
-.12	1.053	1.571	2.636
-.16	1.156	1.716	*

Table 26: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Best Compromise Plan on the parameters β_0=6, β_1=−4, ϕ=1.5 by simulation, sample size n=1000, n=500, n=200, respectively. The number of simulation n_s=6000 and ξ_L=0.7016, ξ_L'=0.703, π=0.333, $SD_{1000} = 44.22$, $SD_{500} = 62.54$, $SD_{200} = 98.89$

	n=1000	n=500	n=200
+.20	124.7	281.1	2646
+.16	81.80	158.7	1053**
+.12	63.71	117.1	1686
+.08	57.72	105.8	1170
+.04	53.90**	105.7	1461
ξ_L'	54.27	103.6**	2.861e^6
-.0014	54.01*	103.7*	2.697$e^{6\star}$
-.04	57.65	120.2	2.963e^6
-.08	61.62	125.3	1.697e^6
-.12	63.23	6593	8.567e^5

Table 27: This table shows the standard deviation of $\widehat{t_{.1}}$ from the Best Compromise Plan on the parameters β_0=6, β_1=−8, ϕ=1.5 by simulation, sample size n=1000, n=500, n=200, respectively. The number of simulation n_s=6000 and ξ_L=0.4625, $\xi_L^{'}$=0.462, π=0.333, $SD_{1000} = 11.53$, $SD_{500} = 16.30$, $SD_{200} = 25.78$

	n=1000	n=500	n=200
+.20	17.30	24.92	40.54
+.16	15.25	21.90	35.24
+.12	13.73	19.50	31.37
+.08	12.59	17.95	28.81
+.04	11.84	16.95	27.33
+0.0005	11.63**	16.72**	27.26**
$\xi_L^{'}$	11.65	16.73	27.30
-.04	12.13	17.49	28.88
-.08	13.19	19.28	32.82
-.12	14.81	21.83	36.61

Table 28: This table show $\widehat{\overline{t_{.1}}}$ at the Optimum Plan on the parameters β_0=4, β_1=−2, ϕ=1 by simulation. The number of simulation n_s=6000 and ξ=0.352, π=0.829.

	n=1000	n=500	n=200
$t_{.1}$	12.97	12.97	12.97
$\widehat{\overline{t_{.1}}}$	13.12	13.24	18.83
b	0.050	0.270	5.860

Table 29: This table show $\widehat{\overline{t_{.1}}}$ at the Optimum Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation. The number of simulation n_s=6000 and ξ=0.215, π=0.854.

	n=1000	n=500	n=200
$t_{.1}$	12.97	12.97	12.97
$\widehat{\overline{t_{.1}}}$	13.00	13.02	13.09
b	0.030	0.050	0.060

Table 30: This table show $\widehat{\overline{t_{.1}}}$ at the Optimum Plan on the parameters β_0=6, β_1=−4, ϕ=1.5 by simulation. The number of simulation n_s=6000 and ξ=0.741, π=0.697.

	n=1000	n=500	n=200
$t_{.1}$	123.1	123.1	123.1
$\widehat{\overline{t_{.1}}}$	132.9	316.1	8932
b	9.80	193	8810

Table 31: This table show $\overline{\widehat{t_{.1}}}$ at the Optimum Plan on the parameters β_0=6, β_1=−8, ϕ=1.5 by simulation. The number of simulation n_s=6000 and ξ=0.455, π=0.730.

	n=1000	n=500	n=200
$t_{.1}$	123.1	123.1	123.1
$\overline{\widehat{t_{.1}}}$	123.7	124.0	125.5
b	0.60	0.90	2.40

Table 32: This table show $\overline{\widehat{t_{.1}}}$ at the Best Compromise Plan on the parameters β_0=4, β_1=−2, ϕ=1 by simulation. The number of simulation n_s=6000 and ξ=0.291, π = 0.333.

	n=1000	n=500	n=200
$t_{.1}$	12.97	12.97	12.97
$\overline{\widehat{t_{.1}}}$	13.24	13.56	14.70
b	0.270	0.590	1.630

Table 33: This table show $\overline{\widehat{t_{.1}}}$ at the Best Compromise Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation. The number of simulation n_s=6000 and ξ=0.232, π = 0.333.

	n=1000	n=500	n=200
$t_{.1}$	12.97	12.97	12.97
$\overline{\widehat{t_{.1}}}$	13.05	13.10	13.24
b	0.080	0.130	0.270

Table 34: This table show $\widehat{\overline{t_{.1}}}$ at the Best Compromise Plan on the parameters β_0=6, β_1=−4, ϕ=1.5 by simulation. The number of simulation n_s=6000 and ξ=0.703, π = 0.333.

	n=1000	n=500	n=200
$t_{.1}$	123.1	123.1	123.1
$\widehat{\overline{t_{.1}}}$	135.6	152.6	$7.203e^4$
b	12.5	29.05	$7.197e^4$

Table 35: This table show $\widehat{\overline{t_{.1}}}$ at the Best Compromise Plan on the parameters β_0=6, β_1=−8, ϕ=1.5 by simulation. The number of simulation n_s=6000 and ξ=0.462, π = 0.333.

	n=1000	n=500	n=200
$t_{.1}$	123.1	123.1	123.1
$\widehat{\overline{t_{.1}}}$	124.4	124.7	126.8
b	1.30	1.60	3.70

Table 36: This table show $\widehat{t_{.1}}$ at the Optimum Plan on the parameters β_0=4, β_1=−2, ϕ=1 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.352, π=0.829, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	4.255	4.229	4.342	3.528	3.759	4.891	3.789	3.668	4.309
$\widehat{\beta_1}$	-2.242	-2.112	-2.026	-1.647	-1.90	-1.926	-2.172	-1.829	-2.054
$\widehat{\phi}$	0.881	0.779	0.70	1.606	1.221	0.303	1.531	1.525	0.746
$\widehat{t_{.1}}$	15.37	13.75	14.27	10.80	11.59	12.68	13.66	12.07	14.44
$\overline{\widehat{t^*_{.1}}}$	15.61	13.92	14.47	10.98	11.75	12.67	21.84	19.02	26.35
$\widehat{b}$	0.240	0.170	0.20	0.180	0.160	-0.010	8.180	6.950	11.91

Table 37: This table show $\widehat{t_{.1}}$ at the Optimum Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.215, π=0.854, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	3.929	3.970	3.985	3.572	3.890	3.873	4.031	3.670	4.021
$\widehat{\beta_1}$	-6.056	-5.982	-5.965	-5.665	-5.904	-5.779	-6.246	-5.817	-5.924
$\widehat{\phi}$	1.163	1.005	1.038	1.619	1.039	1.032	0.987	1.634	1.145
$\widehat{t_{.1}}$	13.32	12.63	13.10	11.34	11.92	11.67	13.27	12.58	14.46
$\overline{\widehat{t^*_{.1}}}$	13.34	12.65	13.12	11.38	11.96	11.71	13.39	12.71	14.61
$\widehat{b}$	0.020	0.020	0.020	0.040	0.040	0.040	0.120	0.130	0.150

Table 38: This table show $\widehat{t_{.1}}$ at the Optimum Plan on the parameters β_0=6, β_1=−4, ϕ=1.5 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.741, π=0.697, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	6.230	6.429	5.864	5.191	6.243	5.583	5.358	5.160	5.311
$\widehat{\beta_1}$	-4.195	-4.401	-3.977	-3.314	-4.416	-3.219	-3.630	-3.179	-3.558
$\widehat{\phi}$	1.405	1.293	1.907	2.347	1.813	1.033	2.537	2.032	2.528
$\widehat{t_{.1}}$	149.2	173.2	122.6	69.42	174.4	64.56	85.19	62.66	81.10
$\overline{\widehat{t^*_{.1}}}$	162.5	188.6	132.8	101.8	751.3	74.37	3134	3035	3145
$\widehat{b}$	13.30	15.40	10.20	32.38	576.9	9.810	3049	2972	3064

Table 39: This table show $\widehat{t_{.1}}$ at the Optimum Plan on the parameters β_0=6, β_1=−8, ϕ=1.5 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.455, π=0.730, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	5.991	6.189	5.951	5.617	6.008	5.714	5.769	5.651	6.187
$\widehat{\beta_1}$	-7.998	-8.161	-7.953	-7.668	-8.057	-7.689	-7.972	-7.662	-8.186
$\widehat{\phi}$	1.539	1.270	1.581	1.994	1.476	1.733	1.873	2.115	1.616
$\widehat{t_{.1}}$	123.9	134.8	120.8	98.02	123.0	100.3	110.4	104.5	154.9
$\overline{\widehat{t^*_{.1}}}$	124.4	135.4	121.3	98.51	123.8	100.8	110.9	105.6	158.7
$\widehat{b}$	0.50	0.60	0.50	0.49	0.80	0.50	0.50	1.10	3.80

Table 40: This table show $\widehat{t_{.1}}$ at the Best Compromise Plan on the parameters β_0=4,β_1=−2, ϕ=1 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.291, π=0.333, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	4.027	4.381	3.761	3.510	4.141	4.408	3.408	3.257	3.841
$\widehat{\beta_1}$	-2.034	-2.220	-1.893	-1.776	-2.352	-1.922	-1.740	-1.380	-1.905
$\widehat{\phi}$	1.036	0.731	1.282	1.672	1.150	0.542	1.595	1.719	1.189
$\widehat{t_{.1}}$	13.64	15.31	11.96	10.85	16.35	12.58	9.546	8.553	12.37
$\overline{\widehat{t^*_{.1}}}$	13.95	15.69	12.18	11.20	17.23	13.20	10.15	9.138	13.91
$\widehat{b}$	0.310	0.380	0.220	0.350	0.880	0.620	0.604	0.585	1.540

Table 41: This table show $\widehat{t_{.1}}$ at the Best Compromise Plan on the parameters β_0=4, β_1=−6, ϕ=1 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.232, π=0.333, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	3.947	4.024	4.058	3.785	4.004	3.742	4.110	3.719	4.274
$\widehat{\beta_1}$	-5.956	-5.999	-6.068	-5.878	-6.110	-5.725	-6.265	-5.721	-6.253
$\widehat{\phi}$	1.034	0.953	0.994	1.193	1.037	1.083	1.0	1.209	0.918
$\widehat{t_{.1}}$	12.57	12.87	13.70	11.72	13.33	10.56	14.48	11.06	16.12
$\overline{\widehat{t^*_{.1}}}$	12.64	12.94	13.77	11.81	13.46	10.64	14.80	11.26	16.54
$\widehat{b}$	0.070	0.070	0.070	0.090	0.130	0.080	0.320	0.20	0.420

Table 42: This table show $\widehat{t_{.1}}$ at the Best Compromise Plan on the parameters β_0=6, β_1=−4, ϕ=1.5 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.703, π=0.333, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	6.169	6.439	5.740	4.952	6.622	6.582	5.148	4.415	5.274
$\widehat{\beta_1}$	-4.146	-4.380	-3.851	-3.134	-4.762	-3.982	-3.432	-2.548	-3.467
$\widehat{\phi}$	1.485	1.284	1.884	2.465	1.605	0.661	2.316	2.770	2.422
$\widehat{t_{.1}}$	144.9	174.2	107.6	55.99	238.4	128.4	66.08	34.52	76.60
$\overline{\widehat{t^*_{.1}}}$	161.9	195.9	117.4	63.99	305.5	170.2	91.21	2308	$5.128e^4$
$\widehat{b}$	17.0	21.70	9.80	8.0	67.10	41.80	25.13	2273	$5.120e^4$

Table 43: This table show $\widehat{t_{.1}}$ at the Best Compromise Plan on the parameters β_0=6, β_1=−8, ϕ=1.5 by simulation. $\widehat{\beta_0}$, $\widehat{\beta_1}$, $\widehat{\phi}$ be the mle of β_0, β_1, ϕ, respectively. The number of bootstrap B=6000 and ξ=0.462, π=0.333, then we can obtain $\overline{\widehat{t^*_{.1}}}$ by Bootstrap

	n=1000			n=500			n=200		
$\widehat{\beta_0}$	5.946	5.974	6.062	5.699	6.003	5.643	5.985	5.621	6.397
$\widehat{\beta_1}$	-7.944	-7.928	-8.081	-7.711	-8.073	-7.598	-8.137	-7.642	-8.344
$\widehat{\phi}$	1.517	1.422	1.517	1.674	1.545	1.602	1.597	1.879	1.291
$\widehat{t_{.1}}$	117.4	116.3	131.9	96.90	125.7	89.41	125.7	95.42	167.6
$\overline{\widehat{t^*_{.1}}}$	118.4	117.3	132.9	97.88	127.2	90.33	128.7	97.33	173.8
$\widehat{b}$	1.0	1.0	1.0	0.980	1.50	0.920	3.0	1.910	6.20