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願將我的所有成果與你們共享。

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# 摘 要

本篇論文的主要目的在探討穩定線性非時變且具備多重輸入時滯系統之$Hankel$範數。 首先，採用穩定等價的觀念來分析時滯系統之穩定性。繼而，針對直饋型輸入時滯系統推導 其$Hankel$運算子與其伴隨運算子，並討論此種運算子之緊緻性。最後，本文分別從誘導範數 的定義及運算子奇異值之最大值即爲其範數的觀念來計算此種運算子之範數。研究結果顯示: 範數之值爲包含延遲時間與時滯個數效應的特定矩陣之行列式的最大零點， 文中並舉例說明$Hankel$範數的計算過程。

# Abstract

*In this thesis, we are concerned with the computation of the Hankel norm for stable linear time-invariant systems with multiple input delays. First of all, the stability of delay systems is investigated by using the concept of stability equivalence. Next, the Hankel operator of linear systems with feedthrough-type input delays and its adjoint are constructed. The compactness of this operator is then examined. Afterward, the norm computation of Hankel operator is studied in two different approaches: one is based on definition of induced operator norm, and the other is based on the fact that the value of operator norm is equal to its largest singular value. The result shows that the Hankel norm is just the largest root of an algebraic equation which actually is the determinant of certain complicated matrix including the effect of time-length and number of delays. Some illustrative examples are presented.*

*Keywords: delay systems, stability, Hankel operator, singular value, feedthrough delay*
*AMS subject: 47B35, 93B60, 93C23, 93D05*

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# Notations

| *Symbol* | *Meaning* |
|---|---|
| $\otimes$ | $A \otimes B$, *Knonecker product of matrices* $A$ *and* $B$ |
| $\oplus$ | $z_1 \oplus z_2$, *direct sum or Knonecker sum of* $z_1$ *and* $z_2$ |
| $\frac{d}{dt}$ | *the differentiation* |
| $\|\cdot\|$ | $\|A\|$, *the induced norm of a matrix* $A$ *which is defined as* $\|A\| = \sup\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$ |
| $<\cdot,\cdot>_H$ | $<u, v>_H$, *the inner product of* $u$ *and* $v$ *on the Hilbert space* $H$ |
| $(A)_{k,k}$ | *the* $(k, k)$ *element of the matrix* $A$ |
| $\mathbb{C}$ | *the set of complex numbers* |
| $\mathbb{C}_+$ | *all complex numbers with real part large than* $0$ |
| $\mathbb{C}_-$ | *all complex numbers with real part less than* $0$ |
| $\mathbf{C} = \mathcal{C}([-T, 0], \mathbb{R}^n)$ | *the Banach space of continuous functions mapping the interval* $[-T, 0]$ *into* $\mathbb{R}^n$ *with the topology of uniform convergence* |
| $\mathcal{D}(T)$ | *domain of the operator* $T$ |
| $\exp,\ e$ | *exponential function* |
| $\|f\|_2$ | $\|f\|_2 = \left(\int_{-\infty}^{\infty} \|f(t)\| dt\right)^{1/2}$ |
| $G(s)$ | *the transfer function of the system* $G$ |
| $\|G\|_H$ | *the Hankel norm of the system* $G$ |
| $H(t)$ | *the Heaviside function* |
| $\mathcal{H}^2(\mathbb{C})$ | *Hardy space of square integrable functions on* $\mathbb{C}_+$ *with values in* $\mathbb{C}$ |
| $\mathcal{H}^\infty$ | *Hardy space of bounded holomorphic functions on* $\mathbb{C}_+$ *with values in* $\mathbb{C}$ |
| $I$ | *the identity matrix* |
| $\mathcal{L}^1(0, \infty)$ | *class of Lebesgue measurable complex-valued functions with* $\int_0^\infty \|f(t)\| dt < \infty$ |

| | |
|---|---|
| $\mathcal{L}^2(j\mathbb{R})$ | *frequency-domain 2-space* $\mathcal{L}^2$ |
| $\mathcal{L}^2 \triangleq \mathcal{L}^2(-\infty,\infty)$ | *class of Lebesgue measurable real-valued matrix functions with* $\int_{-\infty}^{\infty} \Vert f(t)\Vert^2 dt < \infty$ |
| $\mathcal{L}^2_+ \triangleq \mathcal{L}^2[0,\infty)$ | *class of Lebesgue measurable real-valued matrix functions with* $\int_0^{\infty} \Vert f(t)\Vert^2 dt < \infty$ |
| $\mathcal{L}^2_- \triangleq \mathcal{L}^2(-\infty,0]$ | *class of Lebesgue measurable real-valued matrix functions with* $\int_{-\infty}^{0} \Vert f(t)\Vert^2 dt < \infty$ |
| $\mathcal{L}^\infty(j\mathbb{R})$ | $\{G(s) : \Vert G\Vert_\infty < \infty\}$ |
| $\mathcal{P}_p(\Omega;\mathcal{L}(Z_1,Z_2))$ | *functions in* $\mathcal{P}(\Omega;\mathcal{L}(Z_1,Z_2))$ *with* $\int_\Omega \Vert F(t)\Vert^p dt < \infty$ |
| $P_+,\ P_-$ | *orthogonal projection, e.g. if* $y \in \mathcal{L}^2$, *then* $P_+(y) \in \mathcal{L}^2_+,\ P_-(y) \in \mathcal{L}^2_-$ |
| $\mathbb{R}$ | *the set of real numbers* |
| $\mathbb{R}^{n\times m}$ | *space of* $n \times m$ *real matrices* |
| $\mathcal{RH}^\infty,\ \mathcal{RH}^\infty_-$ | *sets of real-rational functions analytic in the close right and left half planes, respectively* |
| *s.t.* | *such that* |
| $\mathbf{U}$ | *the set of all admissible control* |
| $(u)_j$ | *the* $j$ *component of the vector* $u$ |
| $\Gamma$ | *the Hankel operator of the system* $G$ |
| $\Gamma^*$ | *the adjoint of the Hankel operator* $\Gamma$ |
| $\chi_F$ | *simple function defined on set* $F$, $\chi_F(t) = 1$, *if* $t \in F$; $\chi_F(t) = 0$, *otherwise* |
| $\delta x, \delta u, \delta\lambda$ | *the variations to the variables* $x$, $u$ *and* $\lambda$, *respectively* |
| $\delta(t)$ | *the Dirac delta function* |

# Chapter 1

# Introduction

*In many engineering applications, processes are described by complex models which are difficult to analyze and difficult to control. Although reduction of the order of the model overcomes some of these difficulties, it is quite possible to incur a significant loss of accuracy. Therefore, the purpose of model reduction is to replace the plant to be controlled with one of the lower-order possible model while maintaining an overall acceptable degradation in performance. The key of this problem can be stated as follows: given an* $m \times m$ *transfer function* $G \in \mathcal{RH}^\infty$ *with McMillan degree* $n$*, find an* $m \times m$ *transfer function* $\hat{G} \in \mathcal{RH}^\infty$ *with McMillan degree no more than* $k$ *and* $k < n$*, such that* $\|G - \hat{G}\|_H$ *is minimized.*

*The truncated finite dimensional approximation is constructed from the Hankel singular values and vectors which has been studied since 1970's [1]. There are many research results in this problem, e.g. [8, 30, 31], providing good algorithm to construct the approximate model. The approximation of infinite-dimensional linear time-invariant system by lower-order finite-dimensional system is also of great important in the problems of robust control. Most of the results emphasize on delay systems, e.g. [3, 4, 6, 9, 10], which belongs to the class of infinite-dimensional systems but with simple structure such that numerical simulation can be easily conducted.*

*The input delay systems can be characterized according to whether there exist feedthrough terms in output equations or not. For dynamical systems with feedthrough delays, Pandolfi [26, 27] has established the Hankel operators for this type of systems with unit delay time and tried to solve the singular values of this operator. However, it is not succeeded and only a mixed system of differential-difference and differential equations was derived. On the other* *hand, Ohta et. al. [21, 22, 23, 24] have derived the formulas for the singular values and vectors for the systems which contain the input delays in state equation only. However, Yeh and Huang [12, 28] provide a different formula in terms of matrix determinant to compute the Hankel singular values for stable linear systems with single input delay appeared in both state and output equations. In this study, we extend the method of Yeh and Huang to the stable linear systems with multiple input delays.*

*Before computing the Hankel norm, we must consider the stability issue for delay systems. It is well known that the asymptotically behavior of the delay systems is determined from the real part of roots of the characteristic equation [11]. Although there are many researches in the field , e.g. see the survey paper by Kharitonov [19], the stability equivalence concept by Huang [13] is used to study the linear systems with multiple state delays. Delay systems are first transformed into a delay-free one. When the solution of certain matrix equation exists, Lyapunov function from the delay-free system can be selected to be the Lyapunov function for stability analysis of the original system.*

*Once the stability property of delay systems is established, we can compute the Hankel norm of the systems with multiple state and input delays by solving a set of differential-algebraic equation (DAE). Unfortunately, it is very difficult to obtain the closed form solution for the DAE when the state delays present. Thus in this thesis, only the dynamical systems with multiple input delays will be considered for the Hankel norm computation.*

*The following outlines the contents of this thesis. In Chapter* 2*, we introduce mathematical preliminaries and some control concepts for linear time-delay systems, like controllability, observability, transfer function, and the minimum principle, for later use. In Chapter* 3*, the stability in linear systems with multiple delays is studied. Within Chapter* 4*, Hankel operator and its adjoint for linear systems with multiple feedthrough input delays are conducted. In Chapter* 5*, the method to compute Hankel norm, Hankel singular value and vector is established and some numerical examples are used to illustrate the theoretical method. Finally, the conclusion is addressed.*

# Chapter 2

# Mathematical Preliminaries

*Some mathematical relationships and control concept in linear time-delay systems for later use are reviewed here.*

## 2.1 Hankel operators

*Let* $G(s)$ *denote the transfer function of certain dynamical system with* $h(t)$ *being the corresponding impulse response in* $\mathcal{L}^1(0,\infty)$, *i.e.,*

$$G(s) = \int_0^\infty h(t)e^{-st}dt, \qquad s \in \mathbb{C} \tag{2.1}$$

*The associated Hankel operator is defined to be*

$$(\Gamma v)(t) = \int_0^\infty h(t+\tau)v(\tau)d\tau, \quad t > 0 \tag{2.2}$$

*for* $v \in \mathcal{L}^2[0,\infty)$. *We use* $\mathcal{L}^2_+$ *and* $\mathcal{L}^2_-$ *to denote* $\mathcal{L}^2[0,\infty)$ *and* $\mathcal{L}^2(-\infty,0]$, *respectively, for simplicity. Now for any* $G(s) \in \mathcal{L}^\infty(j\mathbb{R})$, *we may define the Hankel operator on* $\mathcal{H}^2(\mathbb{C})$ *by*

$$(\Gamma_G v)(s) = P_+(G(s)v(-s)) \tag{2.3}$$

*for* $v(s) \in \mathcal{H}^2(\mathbb{C})$ *where* $P_+$ *is the orthogonal projection from* $\mathcal{L}^2(j\mathbb{R})$ *onto* $\mathcal{H}^2(\mathbb{C})$. *For* $G \in \mathcal{H}^\infty$, $\Gamma_G$ *is equivalent to* $\Gamma$, *i.e.* $\|G\|_H = \|\Gamma_G\| = \|\Gamma\|$. *By letting* $u(\tau) = v(-\tau)$ *into (2.2), the Hankel operator* $\Gamma$ *could be re-expressed as*

$$\Gamma \ : \ \mathcal{L}^2(-\infty,0] \longrightarrow \mathcal{L}^2[0,\infty) \ : \ u(\cdot) \mapsto y(\cdot) = (\Gamma u)(\cdot)$$

*with*

$$y(t) = (\Gamma u)(t) = \int_{-\infty}^{0} h(t-\tau)u(\tau)d\tau \tag{2.4}$$

*for* $u \in \mathcal{L}_-^2$.

*Some properties related to linear operators are summarized as follows.*

***Definition 2.1*** [5] Let $\mathbf{F}$ be a domain in $\mathbb{C}$, and let $f$ be a function defined on $\mathbf{F}$ with values in $\mathbb{C}$. The function $f$ is *holomorphic* on $\mathbf{F}$ if $\frac{df}{ds}(s_0)$ exists for every $s_0$ in $\mathbf{F}$. The function is said to be *entire* if it is *holomorphic* on $\mathbb{C}$.

***Definition 2.2*** [16] Let $X$ and $Y$ be normed spaces and $T : \mathcal{D}(T) \to Y$ is a linear operator, where $\mathcal{D} \subset X$. The operator is said to be *bounded* if there is a real positive number $c$ such that

$$\|Tx\| \le c\|x\|$$

***Theorem 2.1*** [16] *Let $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D} \subset X$ and $X$ and $Y$ is normed spaces. Then $T$ is continuous if and only if $T$ is bounded.*

***Definition 2.3*** [7] A bounded linear operator $A : X \to Y$, acting between Banach spaces $X$ and $Y$, is called a *Fredholm operator* if its range Im$A$ is closed and the numbers

$$n(A) = \dim\ker A \qquad d(A) = \dim(Y/ImA)$$

are finite.

***Definition 2.4*** [16] Let $X$ and $Y$ be normed spaces. An operator $T : X \to Y$ is called a *compact linear operator* (or *completely continuous linear operator*) if $T$ is linear and if for every bounded subset $\mathcal{M}$ of $X$, the image $T(\mathcal{M})$ is relatively compact, that is, the closure $\overline{T(\mathcal{M})}$ is compact.

## 2.2 Stability of time-delay systems

*Let $\mathbf{C} = \mathcal{C}([-T,0],\mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-T,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Designate the norm of an element $\phi$ in $\mathbf{C}$ by $\|\phi\|_* = \sup_{\tau\in[-T,0]} \|\phi(\tau)\|$ where $\|\cdot\|$ denotes the vector norm in $\mathbb{R}^n$. Suppose*

$f : \mathbb{R} \times \mathbf{C} \to \mathbb{R}^n$ *is continuous and consider the retarded functional differential equation (RFDE)*

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \geq t_0 \\ x(t_0 + \tau) &= \phi(\tau), \quad \forall\, \tau \in [-T, 0] \end{aligned} \tag{2.5}$$

*A general approach to stability analysis is based on the Lyapunov's direct (second) method. There are two different ideal how one can apply this method to delay systems. In the first one, the state of a delay system is defined as the time depending trajectory segment*

$$x_t \ : \ [-T_N, 0] \to \mathbb{R}^n \ : \tau \to x_t(\tau) = x(t + \tau) \tag{2.6}$$

*Here instead of the classical Lyapunov functions it is proposed to use Lyapunov functional (as a function of $x_t$). This approach is called Lyapunov-Krasovskii method. A second idea is based on classical Lyapunov functions and uses a special estimation procedure which allows to exclude delay states in the derivative of the Lyapunov functions. This procedure has been proposed as Lapunov-Razunmikhin method. Those who have interested in the stability analysis of time-delay system can refer to Kharitonov's survey paper [19] and references there in.*

*Let $x(t) = \psi(t, \phi)$ be the solution of RFDE. The definition of stability is discussed as follows.*

***Definition 2.5*** A state $x_e$ is said to be an *equilibrium state* if

$$\psi(t_0, \phi) = x_e \Rightarrow \psi(t, \phi) = x_e, \ \forall t > t_0 \tag{2.7}$$

where $t_0$ is the initial time.

***Definition 2.6*** The equilibrium state $x_e = 0$ is *stable in the sense of Lyapunov* or simply *stable* as $t \to \infty$ if for any positive numbers $t_0$ and $\epsilon$, there exists a $\delta(t_0, \epsilon) > 0$ such that

$$\max_{t_0 \leq t < \infty} \|x(t)\| \leq \epsilon \text{ whenever } \|\phi\|_* \leq \delta$$

***Definition 2.7*** The equilibrium state $x_e = 0$ is *uniformly stable* if for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$

$$\max_{t_0 \leq t < \infty} \|x(t)\| \leq \epsilon \text{ whenever } \|\phi\|_* \leq \delta$$

***Definition 2.8*** The equilibrium state $x_e = 0$ is *asymptotically stable*, if

1. it is stable,

2. every solution which satisfies $\|\phi\|_* \leq \delta$ also satisfies $\lim_{t\to\infty} x(t) = 0$.

***Definition 2.9*** The equilibrium state $x_e = 0$ is *asymptotically stable in the large*, if

1. it is stable,

2. every solution which satisfies $\lim_{t\to\infty} x(t) = 0$, for all arbitrary $\phi(\tau),\ \tau \in [-\hat{T}, 0]$.

*If we use the Lyapunov-Kransovskii method, the relative theorem is stated the following :*

***Theorem 2.2*** [29] *(Lyapunov-Kransovskii Theorem) Suppose $f : \mathbb{R}\times\mathbf{C} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathbf{C}$) into a bounded sets of $\mathbb{R}^n$, and $\alpha, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and nondecreasing functions, $\alpha(s)$ and $\beta(s)$ are positive for $s > 0$, and $\alpha(0) = \beta(0) = 0$. If there is a continuous function $V : \mathbb{R}\times\mathbf{C} \to \mathbb{R}$ such that*

$$\begin{aligned}
&\alpha(\|\phi(0)\|) \leq V(t, \phi) \leq \beta(\|\phi\|_*) \\
&\dot{V}(t, \phi) \leq -\gamma(\|\phi(0)\|)
\end{aligned}$$

*then the solution $x = 0$ of (2.5) is uniformly stable. If $\alpha(s) \to \infty$ as $s \to \infty$, the solution of (2.5) are uniformly bounded. If $\gamma(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.*

*If we use the Lyapunov-Razumikhin method, the relative theorem is stated the following :*

***Theorem 2.3*** *(Lyapunov-Razumikhin Theorem) Suppose $f : \mathbb{R}\times\mathbf{C} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathbf{C}$) into a bounded sets of $\mathbb{R}^n$, and $\alpha, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and nondecreasing functions, $\alpha(s)$ and $\beta(s)$ are positive for $s > 0$, and $\alpha(0) = \beta(0) = 0$. If there is a continuous function $V : \mathbb{R}\times\mathbf{C} \to \mathbb{R}$ such that*

$$\begin{aligned}
&\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|) \\
&\dot{V}(t, \phi) \leq -\gamma(\|x\|) \\
&V(t+\tau, x(t+\tau)) \leq \delta(V(t, x(t)))
\end{aligned}$$

*Here $\delta(s)$ is a continuous function such that $\delta(s) > s$ for $s > 0$. Then the solution $x = 0$ of (2.5) is asymptotically stable.*

*Suppose* $D : \mathbf{C} \to \mathbb{R}^n$ *is a given continuous function and atomic at zero [11], a more general class of neutral delay differential equation (NDDE) is*

$$\begin{aligned} \tfrac{d}{dt}D(x_t) &= f(t, x_t),\ t \geq t_0 \\ x(t_0 + \tau) &= \phi(\tau),\ \forall \tau \in [-T, 0] \end{aligned} \tag{2.8}$$

*The function* $D$ *is called difference operator for NDDE.*

***Definition 2.10*** [29] Suppose $D : \mathbf{C} \to \mathbb{R}^n$ is linear, continuous, and atomic at zero; and let $C_D = \{\phi \in \mathbf{C} : D\phi = 0\}$. The operator $D$ is said to be *stable* if the

$$D(y_t) = 0, \quad t \geq 0, \quad y_0 = \psi \in C_D,$$

is uniformly asymptotically stable.

*In this thesis, we use the following theorem to prove the stability.*

***Theorem 2.4*** [29] *Suppose* $D$ *is stable,* $\alpha(s), \beta(s), \gamma(s) : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous and nondecreasing functions,* $\alpha(s)$ *and* $\beta(s)$ *are positive for* $s > 0$, *and* $\alpha(0) = \beta(0) = \gamma(0) = 0$. *If there is a continuous function* $V : \mathbb{R} \times \mathbf{C} \to \mathbb{R}$ *such that*

$$\begin{aligned} &\alpha(\|D\phi\|) \leq V(t, \phi) \leq \beta(\|\phi\|_*) \\ &\dot{V}(t, \phi) \leq -\gamma(\|D\phi\|) \end{aligned}$$

*then the solution* $x = 0$ *of (2.8) is uniformly stable. If* $\alpha(s) \to \infty$ *as* $s \to \infty$, *the solution of (2.5) are uniformly bounded. If* $\gamma(s) > 0$ *for* $s > 0$, *then the solution* $x = 0$ *is uniformly asymptotically stable.*

## 2.3 Control concepts in linear time-delay systems

*Consider linear time-delay (LTD) systems with the state and output equations of the form:*

$$\sum{}_d : \quad \begin{aligned} \dot{x}(t) &= A(t)x(t) + \sum_{j=1}^{N} A_j(t)x(t - T_{xj}) + B_0(t)u(t) + \sum_{j=1}^{N} B_j(t)u(t - T_{uj}) \\ y(t) &= C(t)x(t) + \sum_{j=1}^{N} C_j(t)x(t - T_{xj}) + D_0(t)u(t) + \sum_{j=1}^{N} D_j(t)u(t - T_{uj}) \end{aligned} \tag{2.9a}$$

*where* $A(t), C(t), B_0(t), D_0(t), A_j(t), B_j(t), C_j(t),$ *and* $D_j(t)$ *are continuous matrix value functions of time, delay time* $T_{xj}$, $T_{uj}$ *are positive constants, with* $T_{x_{j+1}} > T_{x_j}$, $T_{u_{j+1}} > T_{u_j}$, *and $T_{uN} = T_{xN}$, where $j = 1, 2, \cdots, N$. If the system is linear time-invariant (LTI), then the matrices $A, C, B_0, D_0, A_j, B_j, C_j$ and $D_j$ are constant, where $j = 1, 2, \cdots, N$. The initial condition of $\sum_d$ is given by*

$$x_{t_0}(\tau) = \phi(\tau), \quad \tau \in [-\widehat{T}, 0] \tag{2.9b}$$

*where $\phi \in \mathbf{C}$, $\widehat{T} = T_{xN}$.*

*When $u = 0$, the state equation becomes*

$$\dot{x}(t) = A(t)x(t) + \sum_{j=1}^{N} A_j(t)x(t - T_{xj}) \tag{2.10}$$

*with initial condition given by (2.9b) is called the homogeneous state equation or unforced state equation.*

***Definition 2.11*** The state of $\dot{x}(t) = A(t)x(t)$ is *asymptotically stable* on $\mathbb{R}$ if and only if there exist $M > 0$ and $\alpha > 0$ such that

$$\|\Phi(t, t_0)\| \leq Me^{-\alpha(t-t_0)}$$

for all $t \geq t_0$, which $\|\cdot\|$ is matrix norm and

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\tau)\ d\tau\right)$$

*One of the methods of describing a LTI system in the input-output form is using the transfer function. Taking the Laplace transformation on the both sides of given system (2.9a), we get*

$$\begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + \sum_{j=1}^{N} A_j\hat{x}(s)e^{-T_{xj}s} + B_0\hat{u}(t) + \sum_{j=1}^{N} B_j\hat{u}(s)e^{-T_{uj}s} \\ \hat{y}(s) &= C\hat{x}(s) + \sum_{j=1}^{N} C_j\hat{x}(s)e^{-T_{xj}s} + D_0\hat{u}(t) + \sum_{j=1}^{N} D_j\hat{u}(s)e^{-T_{uj}s} \end{aligned}$$

*where $\hat{x}(s), \hat{u}(s)$, and $\hat{y}(s)$ denote the Laplace transformations of the state $x(t)$, input $u(t)$, and output $y(t)$, respectively. Therefore, the system's transfer function (say $G_d(s)$) is given by*

$$G_d(s) = (C + \sum_{j=1}^{N} C_i e^{-T_{xj}s}) \left[ sI - (A + \sum_{j=1}^{N} A_j e^{-T_{xj}s}) \right]^{-1} (B_0 + \sum_{j=1}^{N} B_j e^{-T_{uj}s}) + (D_0 + \sum_{j=1}^{N} D_j e^{-T_{uj}s})$$

*Controllability and observability are two fundamental structural attributes of any control systems. They deal, respectively, with the relationships between the input and the state, and between the state and the output of the system as shown in Figure 2.1. More specifically, their meanings are follows:*

1. *system controllability : does there exist a control u which can transfer the initial state $x_0$ of system to any desired state $x_f$ within a finite period of time $t_f$ ?*
2. *system observability : does the initial state $x_0$ of the system be always identified by observing the output y and input u over a finite time ?*

*The concept of stability and controllability on delay-free system and delay system is different. The effect of delay is shown in Figure 2.2.*

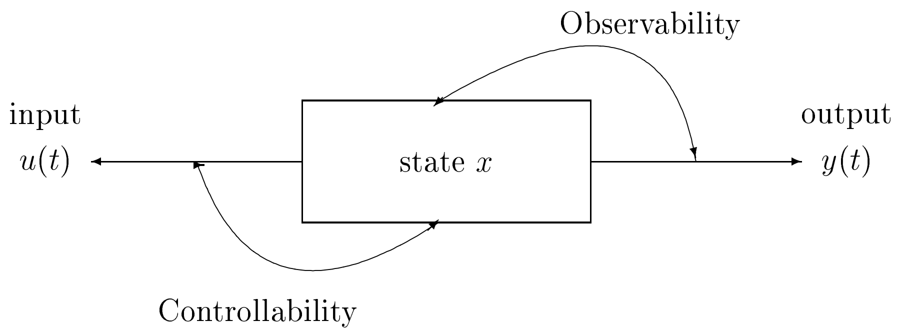



Figure 2.1: Concept for controllability and observability of linear TD systems

*Considering the linear TD system characterized by (2.9a) and (2.9b), if u is measurable and bounded on every finite time interval, it will be called an admissible control.*

*Let the solution of (2.9a) together with (2.9b) is denoted by $x(t, u; t_0, \phi)$ for given control input $u(\tau)$ and initial function $\phi(\tau)$, $\tau \in [t_0 - \widehat{T}, t_1]$. Let $K \subset \mathbf{C}$ and $K_I \subset \mathbf{C}$.*

***Definition 2.12*** System given in (2.9a) and (2.9b) is *controllable to a function $\alpha(\cdot) \in K$ w.r.t. the space of initial function $K_I$* if for any given initial function $\phi(\cdot) \in K_I$, there exists a finite time $t_1 > t_0$, and an admissible control $u(t)$, $t \in [t_0 - \widehat{T}, t_1]$ s.t.

$$x(t_1 + \tau, u; t_0, \phi) = \alpha(\tau), \quad \tau \in [-\widehat{T}, 0]$$

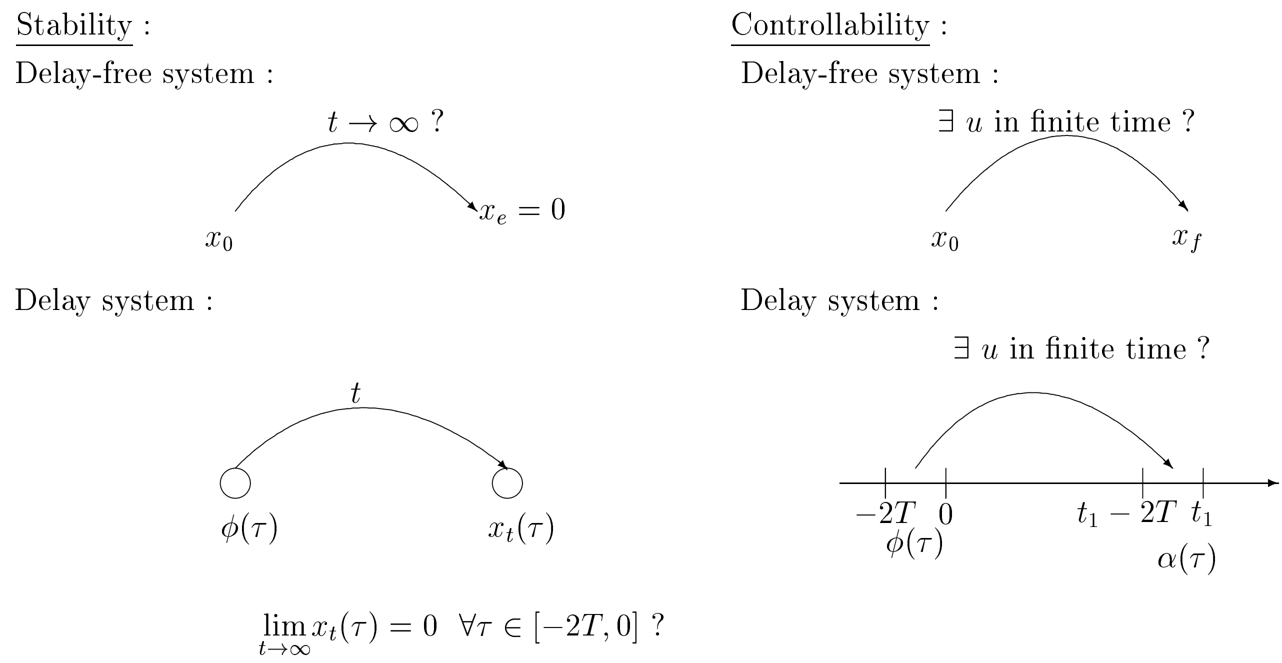



Figure 2.2: The effect of delay on stability and controllability

***Definition 2.13*** If system (2.9) is controllable to all functions in $K$, then it is said to be controllable to the space $K$.

***Definition 2.14*** If $\alpha(\cdot) \equiv 0$, then the system is said to be *controllable to the origin*, or *fixed-time completely controllable.*

***Definition 2.15*** If $t_1$ is constant, the corresponding type of controllability is said to be *uniform.*

*For linear delay-free system, the controllability to the origin is equivalent to the controllability to any function. However, in general, this is not true in the case of linear TD systems.*

***Definition 2.16*** The system given in (2.9a) and (2.9b) is *observable* in $[t_0, t_1]$ if the initial function $\phi(\cdot) \in \Sigma$ in $[t_0 - \widehat{T}, t_0]$ can be uniquely determined from the knowledge of the control $u(\cdot)$ over $[t_0 - T, t_1]$ and observation $y(\cdot)$ over $[t_0, t_1]$.

Those who are interested in the controllability and observability criteria can refer to [17] for more detail discussion.

*In this thesis, we concern about the system with some state and input delay times which are commensurate, i.e.* $T_{xj} = T_{uj} = jT$, *for simplicity.*

## 2.4 The minimum principle for time-delay systems

*Since the matrix* $D_0$ *does not affect the value of Hankel norm, we let* $D_0 \equiv 0$. *Thus the system for studying Hankel norm computation is given by*

$$\sum : \begin{cases} \dot{x}(t) = Ax(t) + B_0 u(t) + \sum_{j=1}^{N} B_j u(t - jT), \ x(-\infty) = 0 \\ y(t) = Cx(t) + \sum_{j=1}^{N} D_j u(t - jT), \ t \in (-\infty, \infty) \end{cases} \tag{2.11}$$

*The variational principle is one of the method to compute the Hankel norm of the system* $\sum$. *We establish the minimum principle for time delay systems here. Let us consider a nonlinear dynamical system with commensurate input delays :*

$$\dot{x}(t) = f(x(t), x(t - T), \ldots, x(t - NT), u(t), u(t - T), \ldots, u(t - NT), t) \tag{2.12}$$

*with initial state and control functions given by*

$$x(t_0+\tau)=\phi(\tau),\quad u(t_0+\tau)=\phi_u(\tau),\quad -NT\le\tau\le 0 \tag{2.13}$$

*where $t_0$ is the initial time, and $\phi(\tau)$ and $\phi_u(\tau)$ are continuous functions on $[-NT,0]$.*

*Let the positive cost function with fixed terminal time $t_f$ be defined by*

$$J(x,u)=\int_{t_0}^{t_f} F(x(t),x(t-T),\ldots,x(t-NT),u(t),u(t-T),\ldots,u(t-NT),t)dt \tag{2.14}$$

*where $F(\cdot)$ is the cost function's intergrand. Let the set of all admissible control be denoted by $\mathbf{U}$. The object of the optimal control is to find an optimal control function $u_*\in\mathbf{U}$ which satisfies the dynamic constraint (2.12) associated with (2.13) while minimizing the cost function (2.14).*

*This optimization problem is solved by studying second-order variations of $J$ resulting from the first-order variations in $u$ and $x$, and requiring that these be zero at optimum. The necessity condition for this optimum is stated as follows.*

***Theorem 2.5*** *Consider the nonlinear dynamical system given in (2.12). We define a Hamiltonian function $H=F+\lambda^T f$. Suppose the optimal value of the cost function is given by*

$$J(x_*,u_*)=\min_{u\in\mathbf{U}} J(x,u) \tag{2.15}$$

*Then the optimal control $u_*$, optimal state $x_*$, and $\lambda$ are solved by the following equations :*

*1.*

$$\dot{\lambda}=\begin{cases} -\dfrac{\partial H}{\partial x}-\displaystyle\sum_{i=1}^{N}\left.\dfrac{\partial H}{\partial x_{id}}\right|_{t=t+iT}, & t_0<t<t_f-NT \\ -\dfrac{\partial H}{\partial x}-\displaystyle\sum_{i=1}^{j}\left.\dfrac{\partial H}{\partial x_{id}}\right|_{t=t+iT}, & t_f-(j+1)T<t<t_f-jT,\ j=1,2,\cdots,N-1 \\ -\dfrac{\partial H}{\partial x}, & t_f-T<t<t_f \end{cases} \tag{2.16}$$

*2.*

$$0 = \begin{cases} -\dfrac{\partial H}{\partial u} - \displaystyle\sum_{i=1}^{N} \dfrac{\partial H}{\partial u_{id}}\Bigg|_{t=t+iT}, & t_0 < t < t_f - NT \\ -\dfrac{\partial H}{\partial u} - \displaystyle\sum_{i=1}^{j} \dfrac{\partial H}{\partial u_{id}}\Bigg|_{t=t+iT}, & t_f - (j+1)T < t < t_f - jT,\ j = 1, 2, \cdots, N-1 \\ -\dfrac{\partial H}{\partial u}, & t_f - T < t < t_f \end{cases} \tag{2.17}$$

*3.*

$$\dot{x}(t) = f(x(t), x(t-T), \ldots, x(t-NT), u(t), u(t-T), \ldots, u(t-NT), t) \tag{2.18}$$

*where $x_{id} \triangleq x(t-iT), u_{id} \triangleq u(t-iT)$, $i = 1, 2, \cdots, N$, and $\lambda(t)$ is the Lagrange multiplier satisfying $\lambda(t_f) = 0$. And the second variation of $J$ on $(x_*, u_*)$ must be positive, i.e. $\delta^2\ J(x_*, u_*) > 0$ or equivalently, the corresponding Henssian matrix of $H$ must be positive definite, i.e.*

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \cdots & \frac{\partial^2 H}{\partial x \partial x_{Nd}} & \frac{\partial^2 H}{\partial x \partial u} & \cdots & \frac{\partial^2 H}{\partial x \partial u_{Nd}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial x_{Nd} \partial x} & \cdots & \frac{\partial^2 H}{\partial x_{Nd}^2} & \frac{\partial^2 H}{\partial x_{Nd} \partial u} & \cdots & \frac{\partial^2 H}{\partial x_{Nd} \partial u_{Nd}} \\ \frac{\partial^2 H}{\partial u \partial x} & \cdots & \frac{\partial^2 H}{\partial u \partial x_{Nd}} & \frac{\partial^2 H}{\partial u^2} & \cdots & \frac{\partial^2 H}{\partial u \partial u_{Nd}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial u_{Nd} \partial x} & \cdots & \frac{\partial^2 H}{\partial u_{Nd} \partial x_{Nd}} & \frac{\partial^2 H}{\partial u_{Nd} \partial u} & \cdots & \frac{\partial^2 H}{\partial u_{Nd}^2} \end{bmatrix} > 0 \tag{2.19}$$

***Proof.*** *Following the concept of Lagrange multipliers in constrained optimization problem, let $\lambda(t)$ be an $n$-vector function seeking to minimize, with respect to $u(\cdot)$ and $x(\cdot)$, augmented cost function*

$$\begin{aligned} J_a(x, u, \lambda) &= \int_{t_0}^{t_f} (H - \lambda^T \dot{x}) dt \\ &= \int_{t_0}^{t_f} (F + \lambda^T f - \lambda^T \dot{x}) dt \end{aligned}$$

*Let $\delta x, \delta u$ denote the variations from the optimal trajectory and control, i.e. $x \to x_* + \alpha\delta x$, $u \to u_* + \alpha\delta u$ for some constant $\alpha$. And the corresponding delay terms in $x$ and $u$ are also variant, i.e. $x_{id} \to x_{*id} + \alpha\delta x_{id}$, $u_{id} \to u_{*id} + \alpha\delta u_{id}$, $i = 1, 2, \ldots, N$. Then the first variation* *of $J_a$ is computed as follows.*

$$\begin{aligned}
\delta J_a(x,u,\lambda) &= \frac{d}{d\alpha} J_a(x+\alpha\delta x, u+\alpha\delta u,\lambda)\Big|_{\alpha=0} \\
&= \int_{t_0}^{t_f} \left[ \delta x^T \frac{\partial H}{\partial x} + \delta u^T \frac{\partial H}{\partial u} + \sum_{i=1}^{N} \delta x_{id}^T \frac{\partial H}{\partial x_{id}} + \sum_{i=1}^{N} \delta u_{id}^T \frac{\partial H}{\partial u_{id}} \right] dt - \int_{t_0}^{t_f} \lambda^T \delta \dot{x} dt \\
&= \int_{t_0}^{t_f} \delta x^T \left[ \frac{\partial H}{\partial x} + \dot{\lambda} \right] dt + \sum_{i=1}^{N} \int_{t_0-iT}^{t_f-iT} \delta x^T \left[ \frac{\partial H}{\partial x_{id}} \right]_{t=t+iT} dt + \\
& \int_{t_0}^{t_f} \delta^T u \left[ \frac{\partial H}{\partial u} \right] dt + \sum_{i=1}^{N} \int_{t_0-iT}^{t_f-iT} \delta u^T \left[ \frac{\partial H}{\partial u_{id}} \right]_{t=t+iT} dt + \lambda^T \delta x\Big|_{t_0}^{t_f}
\end{aligned}$$

*Since $x(t)$ and $u(t)$, for $t \in [t_0 - NT, t_0]$ are prescribed in (2.13), therefore the corresponding variations $\delta x$ and $\delta u$ on this interval should be zero. The optimal condition is obtained by letting $\delta J_a(x_*, u_*, \lambda) = 0$. Thus the optimal $x_*, u_*$, and $\lambda$ must satisfy (2.16),(2.17), and (2.19), with $\lambda(t_f) = 0$. The variations $\delta u$ and $\delta x$ must also satisfy*

$$\delta \dot{x} = f(\delta x, \delta x_{1d}, \ldots, \delta x_{Nd}, \delta u, \delta u_{1d}, \ldots, \delta u_{Nd}, t)$$

*with initial conditions*

$$\delta x(t) = 0, \delta u(t) = 0, \;\; t_0 - NT \le t \le t_0$$

*The second variation of $J$ is given by*

$$\delta^2 J_a(x,u,\lambda) = \int_{t_0}^{t_f} \begin{bmatrix} \delta x^T & \cdots & \delta x_{Nd}^T & \delta u^T & \cdots & \delta u_{Nd}^T \end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 H}{\partial x^2} & \cdots & \frac{\partial^2 H}{\partial x \partial x_{Nd}} & \frac{\partial^2 H}{\partial x \partial u} & \cdots & \frac{\partial^2 H}{\partial x \partial u_{Nd}} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 H}{\partial x_{Nd} \partial x} & \cdots & \frac{\partial^2 H}{\partial x_{Nd}^2} & \frac{\partial^2 H}{\partial x_{Nd} \partial u} & \cdots & \frac{\partial^2 H}{\partial x_{Nd} \partial u_{Nd}} \\
\frac{\partial^2 H}{\partial u \partial x} & \cdots & \frac{\partial^2 H}{\partial u \partial x_{Nd}} & \frac{\partial^2 H}{\partial u^2} & \cdots & \frac{\partial^2 H}{\partial u \partial u_{Nd}} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 H}{\partial u_{Nd} \partial x} & \cdots & \frac{\partial^2 H}{\partial u_{Nd} \partial x_{Nd}} & \frac{\partial^2 H}{\partial u_{Nd} \partial u} & \cdots & \frac{\partial^2 H}{\partial u_{Nd}^2}
\end{bmatrix}
\begin{bmatrix} \delta x \\ \vdots \\ \delta x_{Nd} \\ \delta u \\ \vdots \\ \delta u_{Nd} \end{bmatrix} dt$$

*In order to have a minimal value for $J$ at $(x_*, u_*)$, we need*

$$\delta^2 J_a(x_*, u_*, \lambda) = \delta^2 J(x_*, u_*) > 0$$

*i.e. Henssian matrix of H on $(x_*, u_*)$ must be positive definite, i.e.*

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \cdots & \frac{\partial^2 H}{\partial x \partial x_{Nd}} & \frac{\partial^2 H}{\partial x \partial u} & \cdots & \frac{\partial^2 H}{\partial x \partial u_{Nd}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial x_{Nd} \partial x} & \cdots & \frac{\partial^2 H}{\partial x_{Nd}^2} & \frac{\partial^2 H}{\partial x_{Nd} \partial u} & \cdots & \frac{\partial^2 H}{\partial x_{Nd} \partial u_{Nd}} \\ \frac{\partial^2 H}{\partial u \partial x} & \cdots & \frac{\partial^2 H}{\partial u \partial x_{Nd}} & \frac{\partial^2 H}{\partial u^2} & \cdots & \frac{\partial^2 H}{\partial u \partial u_{Nd}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial^2 H}{\partial u_{Nd} \partial x} & \cdots & \frac{\partial^2 H}{\partial u_{Nd} \partial x_{Nd}} & \frac{\partial^2 H}{\partial u_{Nd} \partial u} & \cdots & \frac{\partial^2 H}{\partial u_{Nd}^2} \end{bmatrix} > 0$$

■

# Chapter 3

# Stability in Linear Time-Delay Systems

*This chapter is devoted to the study of the delays on the asymptotic behavior of the solution of linear retarded differential difference equation. A special case of the retarded equation considered here is*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{N} A_j x(t - jT) \\ x_{t_0}(\tau) &= \phi(\tau), \quad \tau \in [-NT, 0] \end{aligned} \tag{3.1}$$

*where* $x \in \mathbb{R}^n$, $T > 0$, *and* $A$ *and each* $A_j$ *is an* $n \times n$ *matrix,* $j = 1, 2, \cdots, N$.

*It is well known that the asymptotical behavior of the solutions is determined from the roots of the characteristic function*

$$p(s) \triangleq \det[sI - A - \sum_{j=1}^{N} A_j e^{-sjT}]$$

*The zero solution of (3.1) is uniformly asmptotically stable if and only if all roots* $p(s)$ *lie in the open left half complex plane (see, for example, [11]). Since* $p(s)$ *is a transcendental polynomial containing the term* $e^{-sjT}$, *it is every difficult to apply this criteria to check the stability. Our primary object is to given condition on the coefficients* $A$, $A_j$ *in (3.1) which will ensure the system (3.1) is asymptotically stable.*

*Unlike the research idea presented in Section 2.2, a stability equivalence between linear delay and non-delay systems is examined. Once the solution of a certain matrix equation exists, we can transform a delay system into a non-delay one. The Lyapunov function from the non-delay system is selected as our Lyapunov functional of the original delay system for stability analysis. Our results show that delay system is stable if and only if all equivalent systems without delay are stable. On the other hand, when the existence of solution of certain matrix equation fail, we can apply the existing results from other research papers, e.g.[19] for stability analysis.*

## 3.1 Single time-delay case

*The linear time-invariant system with single state delay is considered first:*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-T), \\ x_{t_0}(\tau) &= \phi(\tau), \quad -T \leq \tau \leq 0 \end{aligned} \tag{3.2}$$

*Define a new state variable*

$$z(t) \triangleq x_t(0) + \Theta(t) \star x_t = x(t) + \Theta(t) \star x_t \tag{3.3}$$

*where*

$$\Theta(t) \star x_t \triangleq \int_{-T}^{0} \Theta(-\tau) x_t(\tau) d\tau = \int_{-T}^{0} \Theta(-\tau) x(t+\tau) d\tau \tag{3.4}$$

*By change of variable,* $\Theta(t) \star x_t$ *can be expressed as*

$$\Theta(t) \star x_t = \int_{t-T}^{t} \Theta(t-\tau) x(\tau) d\tau$$

*and its derivative with respect to* $t$ *is given by*

$$\begin{aligned} \frac{d}{dt}\Theta(t) \star x_t &= \Theta(t-\tau)x(\tau)|_{t-T}^{t} + \int_{t-T}^{t} \dot{\Theta}(t-\tau)x(\tau)d\tau \\ &= \Theta(0)x(t) - \Theta(T)x(t-T) + \int_{t-T}^{t} \dot{\Theta}(t-\tau)x(\tau)d\tau \end{aligned}$$

*Then the differentiation of* $z(t)$ *is computed by*

$$\begin{aligned} \dot{z}(t) &= \dot{x}(t) + \frac{d}{dt}\Theta(t) \star x_t \\ &= [A + \Theta(0)]x(t) + [A_1 - \Theta(T)]x(t-T) + \int_{t-T}^{t} \dot{\Theta}(t-\tau)x(\tau)d\tau \end{aligned} \tag{3.5}$$

*From (3.5), it is noted that different choice of* $\Theta(t)$ *may transform the system (3.2) into different type of system. If possible, we have better to keep the new system delay free. Firstly, if we choose* $\Theta(t)$ *as the solution of*

$$\begin{aligned} \dot{\Theta}(t) &= [A + \Theta(0)]\Theta(t) \\ \Theta(T) &= A_1 \end{aligned} \tag{3.6}$$

*then (3.5) becomes*

$$\begin{aligned} \dot{z}(t) &= [A + \Theta(0)]x(t) + \int_{t-T}^{t} [A + \Theta(0)]\Theta(t-\tau)x(\tau)d\tau \\ &= [A + \Theta(0)][x(t) + \Theta(t) \star x_t] \\ &= [A + \Theta(0)]z(t) \end{aligned}$$

*with the initial state defined by*

$$\begin{aligned} z_0 &= x(t_0) + \Theta(t_0) \star x_t \\ &= \phi(0) + \int_0^T \Theta(\tau)\phi(-\tau)d\tau \end{aligned}$$

*Here we arrive a new system :*

$$\begin{aligned} \dot{z}(t) &= [A + \Theta(0)]z(t), \\ z(t_0) &= z_0 = \phi(0) + \int_0^T \Theta(\tau)\phi(-\tau)d\tau \end{aligned} \tag{3.7}$$

*provided that the solution* $\Theta(t)$ *of (3.6) exists. Observe that if the solution of (3.2),* $x(t) \to 0$ *as* $t \to \infty$*, it follows from (3.3) that* $z(t) \to 0$ *as* $t \to \infty$ *also; and if* $z(t) \to 0$ *as* $t \to \infty$ *and* $D(x_t) = 0$ *is asymptotically stable, then* $x(t) \to 0$ *as* $t \to \infty$*. Thus, we called the system (3.7) as the stability equivalence system of the system (3.2). Since equation (3.6) may possible have more than one solution, it means we could obtain many equivalent systems like (3.7) corresponding the given system (3.2). Therefore the system (3.2) is stable if and only if all the equivalent systems with asympototically stable* $D(x_t) = 0$ *are stable.*

*On the other hand, if there is no solution* $\Theta(t)$ *satisfying (3.6), a good choice for* $\Theta(t)$ *is*

$$\dot{\Theta}(t) = 0, \quad \Theta(t) = A_1 \tag{3.8}$$

*Then the rate of change of* $z(t)$ *from (3.5) becomes*

$$\dot{z}(t) = (A + A_1)x(t)$$

*with initial condition of* $z(t)$ *as*

$$z_0 = \phi(0) + \int_{-T}^{0} A_1 \phi(\tau) d\tau$$

*Again, an alternative system is obtained :*

$$\begin{aligned} \dot{z}(t) &= [A + A_1]x(t), \\ z(t_0) &= z_0 = \phi(0) + A_1 \int_{-T}^{0} \phi(\tau) d\tau \end{aligned} \tag{3.9}$$

*which solution is given by*

$$z(t) = z_0 + (A + A_1) \int_{t_0}^{t} x(\tau) d\tau \tag{3.10}$$

*When* $z(t) \to 0$ *as* $t \to \infty$, *in order to let the improper integral in (3.10) exist, it must hold* $x(t) \to 0$ *as* $t \to \infty$. *If* $x(t) \to 0$ *as* $t \to \infty$, *from (3.3), it follows that* $z(t) \to 0$ *as* $t \to \infty$.

*From above discussion, the weighted energy of* $z(t)$, *i.e.,*

$$V(z(t)) = z(t)^T P z(t), \quad P > 0$$

*is a good candidate of Lyapunov function of the system (3.2) for stability analysis.*

## 3.2 Multiple time-delay case

*By using the same equivalence concept developed in Section* 3.1, *the equivalence state variable* $z(t)$ *is defined the same as in equation (3.3), i.e.,*

$$\begin{aligned} z(t) &= D(x_t) \triangleq x_t(0) + \Theta(t) \star x_t \\ &= x(t) + \Theta(t) \star x_t \end{aligned} \tag{3.11}$$

*where* $\Theta(t)$ *is now defined by*

$$\Theta(t) \triangleq \sum_{j=1}^{N} \Theta_j(t) \chi_{[0,jT]}(t)$$

*with*$\chi_F$ *be a simple function acting on* $F$, *i.e.*

$$\chi_F(t) = \begin{cases} 1, & t \in F \\ 0, & t \notin F \end{cases}$$

*The convolution integral $\Theta(t) \star x_t$ is expressed as*

$$\begin{aligned}\Theta(t) \star x_t &= \sum_{j=1}^{N} \int_{-jT}^{0} \Theta_j(-\tau) x_t(\tau) d\tau \\ &= \sum_{j=1}^{N} \int_{t-jT}^{t} \Theta_j(t-\tau) x(\tau) d\tau\end{aligned}$$

*And we note that $\Theta(0)$ is given by*

$$\Theta(0) = \sum_{j=1}^{N} \Theta_j(0) = \Theta_1(0) + \Theta_2(0) + \cdots + \Theta_N(0) \tag{3.12}$$

*Then the stability equivalence system of (3.11) is given by*

$$\begin{aligned}\dot{z}(t) &= [A + \Theta(0)] z(t), \quad t > t_0 \\ z(t_0) &= z_0 = \phi(0) + \sum_{j=1}^{N} \int_{0}^{jT} \Theta_j(\tau) \phi(-\tau) d\tau\end{aligned} \tag{3.13}$$

*provided that there exist solutions $\Theta_j(t)$ for the following matrix differential equation:*

$$\begin{cases} \dot{\Theta}_j(\tau) = (A + \Theta(0)) \Theta_j(\tau), \text{ with } \Theta_j(jT) = A_j \text{ for } \tau \in [0, jT] \\ \Theta_j(\tau) = 0, \qquad \text{elsewhere} \end{cases} \tag{3.14}$$

*with $j = 1, 2, \ldots, N$. The solution $\Theta_j(t)$ of (3.14) is given by*

$$\Theta_j(\tau) = \exp\left[(A + \Theta(0))\tau\right] \Theta_j(0)$$

*and hence $\Theta_j(0)$ must be the solution of the matrix equation:*

$$\exp\left[(A + \Theta(0)) jT\right] \Theta_j(0) = A_j, \quad j = 1, 2, \ldots, N \tag{3.15}$$

*In order to prove that the system (3.13) is stability equivalence to the system (3.11), the following lemma needed.*

***Lemma 3.1*** [2] *Suppose $A$ is stable. Then there exist a positive definite, Hermitian matrix $P$ such that*

$$PA + A^T P + R = 0 \tag{3.16}$$

*where $R$ is any positive definite Hermitian matrix.*

***Lemma 3.2*** *Let $\Theta_j(\tau)$ is the solution of (3.14) and $\Theta(0)$ is defined by (3.12). Suppose that $A + \Theta(0)$ is asymptotically stable and if $D(x_t) = x(t) + \Theta(t) \star x(t) = 0, \ \forall \ t > t_0$ is asymptotically stable about $x = 0$. Let $V(\cdot)$ be a scalar function defined by $V(t, x_t) = D(x_t)^T P D(x_t)$. Then there exist continuous and nondecreasing function $\alpha(s), \beta(s), \gamma(s) : \mathbb{R}^+ \to \mathbb{R}^+$*

$$\begin{aligned} &\alpha(\|D\phi\|) \leq V(t, \phi) \leq \beta(\|\phi\|_*) \\ &\dot{V}(t, \phi) \leq -\gamma(\|D(\phi)\|) \end{aligned}$$

*and $\alpha(s) \to \infty$ as $s \to \infty$, and $\gamma(s) > 0$ for $s > 0$.*

***Proof.*** *Since $A + \Theta(0)$ is stable, then for any $R > 0$, by Lemma 3.2, there exists $P > 0$ such that*

$$P(A + \Theta(0)) + (A + \Theta(0))^T P + R = 0$$

*The total derivative of Lyapunov functional $V(t, x_t)$ is given by*

$$\begin{aligned} \dot{V}(t, x_t) &= D(x_t)^T \left( P(A + \Theta(0)) + (A + \Theta(0))^T P \right) D(x_t) \\ &= -D(x_t)^T R D(x_t) < 0 \end{aligned}$$

*Since $P$ and $R$ are positive definite, then*

$$\begin{aligned} V(t, x_t) &= D(x_t)^T P^{\frac{1}{2}} P^{\frac{1}{2}} D(x_t) = \|P^{\frac{1}{2}} D(x_t)\|^2 \\ \dot{V}(t, x_t) &= -D(x_t)^T R^{\frac{1}{2}} R^{\frac{1}{2}} D(x_t) = -\|R^{\frac{1}{2}} D(x_t)\|^2 \end{aligned}$$

*We obtain*

$$\lambda_{min}(P)\|D(x_t)\|^2 \leq V(t, x_t) \leq \lambda_{max}(P)\|D(x_t)\|^2$$

*Since*

$$\begin{aligned} \|D(x_t)\| &= \|x(t) + \Theta(t) \star x_t\| \\ &= \left\| x(t) + \sum_{j=1}^{N} \int_{t-jT}^{t} \Theta_j(t-\tau) x(\tau) d\tau \right\| \\ &\leq \left\| x(t)\| + \sum_{j=1}^{N} \| \int_{t-jT}^{t} \Theta_j(t-\tau) x(\tau) d\tau \right\| \end{aligned}$$

*By Mean value property of integrals, we obtain*

$$\|D(x_t)\| \leq \left( I + \sum_{j=1}^{N} \| \int_{t-jT}^{t} \Theta_j(t-\tau) d\tau \| \right) \|x_t\|_* \tag{3.17}$$

*i.e.,*

$$\lambda_{min}(P)\|D(\phi)\|^2 \leq V(t,\phi) \leq \lambda_{max}(P)\|D(\phi)\|^2 \leq \lambda_{max}(P)\left(I+\sum_{j=1}^{N}\|\int_{t-jT}^{t}\Theta_j(t-\tau)d\tau\|\right)^2 \|\phi\|_*^2$$

*and*

$$\dot{V}(t,\phi) \leq -\lambda_{min}(R)\|D(\phi)\|^2$$

*Choose*

$$\begin{aligned}\alpha(s) &= \lambda_{\min}(P)s^2\\ \beta(s) &= \lambda_{\max}(P)\left(I+\sum_{j=1}^{N}\|\int_{t-jT}^{t}\Theta_j(t-\tau)d\tau\|\right)^2 s^2\\ \gamma(s) &= \lambda_{\min}(R)s^2\end{aligned}$$

*Then* $\alpha(s)\to\infty$ *as* $s\to\infty$ *and* $\gamma(s)>0$ *for* $s>0$. ■

***Theorem 3.1*** *Suppose that there is a smooth nonsingular solution* $\Theta_j(t)$ *of (3.14) and the TD system be described by (3.1) is stable if and only if the equivalent linear dynamical system (3.13) is asymptotically stable and* $D(x_t) = x(t)+\Theta(t)\star x_t = 0,\ \forall\ t>t_0$ *is asymptotically stable.*

***Proof. (Sufficiency)*** *Before proving , we must show that the operator D is stable. Firstly, we want to show that the operator is linear. Since* $\Theta(t)\star x_t$ *is linear, the operator D satisfies the property. Secondly, we want to show that the operator D is continuous. By the equation (3.17), we obtain the operator D is bounded. And by Theorem 2.1, we obtain D is continuous. Thirdly, D is atomic at zero. And since* $D(x_t)=0,\ \forall\ t>t_0$ *is asymptotically stable, by Definition 2.9, D is stable. Now, suppose that there is a smooth solution* $\Theta_j(t)$ *of (3.13) such that* $A+\Theta(0)$ *is stable. By Lemma 3.1, 3.2 and Theorem 2.4, we know (3.1) is asymptotically stable.*

***(Necessity)*** *Suppose(3.1) is stable, then associated characteristic equation*

$$\det\left(sI-A-\sum_{j=1}^{N}A_j e^{-sjT}\right)=0$$

*has all roots possessing positive real part. By (3.15) we have*

$$\begin{aligned}0 &= \det\left\{sI-A-\sum_{j=1}^{N}\exp\left[(A+\Theta(0)-sI)\,jT\right]\Theta_j(0)\right\}\\ &= \det\left\{sI-A-\Theta(0)+\Theta(0)-\sum_{j=1}^{N}\exp\left[(A+\Theta(0)-sI)\,jT\right]\Theta_j(0)\right\}\end{aligned}$$

*Let $L$ denote $sI - (A + \Theta(0))$. Expanding the exponential term in power series given as*

$$
\begin{aligned}
0 &= \det\left\{L + \Theta(0) - \sum_{j=1}^{N} e^{-jLT}\Theta_j(0)\right\} \\
&= \det\left\{L + \Theta(0) - \sum_{j=1}^{N}\left(I - jLT + \frac{(jLT)^2}{2!} + \cdots + (-1)^n\frac{(jLT)^n}{n!} + \cdots\right)\Theta_j(0)\right\} \\
&= \det\left\{L + \sum_{j=1}^{N}\left(jLT\Theta_j(0) - \frac{(jLT)^2}{2!}\Theta_j(0) + \cdots + (-1)^{n-1}\frac{(jLT)^n}{n!}\Theta_j(0) + \cdots\right)\right\} \\
&= \det(L)\det\left\{I + \sum_{j=1}^{N}\left(jT\Theta_j(0) - \frac{L(jT)^2}{2!}\Theta_j(0) + \cdots + (-1)^{n-1}\frac{L^{n-1}(jT)^n}{n!}\Theta_j(0) + \cdots\right)\right\} \\
&= \det(L)\det\left\{I + \sum_{j=1}^{N}\left(I - \frac{jLT}{2!} + \cdots + (-1)^{n-1}\frac{(jLT)^{n-1}}{n!} + \cdots\right) jT\Theta_j(0)\right\} \\
&= \det(L)\det\left\{I + \sum_{j=1}^{N}\left(\sum_{n=0}^{\infty}(-1)^n\frac{(jLT)^n}{(n+1)!}\right) jT\Theta_j(0)\right\}
\end{aligned}
$$

*Since*

$$\|(-1)^n\frac{(jLT)^n}{(n+1)!}\| \le \|\frac{(jLT)^n}{n!}\|$$

*it following that*

$$\|\sum_{n=0}^{\infty}(-1)^n\frac{(jLT)^n}{(n+1)!}\| \le \sum_{n=0}^{\infty}\|\frac{(jLT)^n}{n!}\| = e^{\|L\|jT}, \quad \forall\ s$$

*By the Weierstrass M-test, we have*

$$\sum_{n=0}^{\infty}(-1)^n\frac{(jLT)^n}{(n+1)!} \quad \text{converges uniformly}$$

*Since* $\sum_{n=0}^{\infty}(-1)^n\frac{(jLT)^n}{(n+1)!}$ *converges to an entire function. Then the eigenvalues* $\lambda_i\left(A + \Theta(0)\right)$ *are contained in the spectrum of the matrix* $A + \sum_{j=1}^{N} A_j e^{-sjT}$. *Therefore*

$$\lambda_i\left(A + \Theta(0)\right) < 0, \quad \forall\ i$$

■

*Observe the necessary part of the proof in the Theorem 3.1, we know the eigenvalues* $\lambda_i\left(A + \Theta(0)\right)$ *are contained in the spectrum of the matrix* $A + \sum_{j=1}^{N} A_j e^{-sjT}$.

***Corollary 3.1*** *Assume $\Theta(t)$ exists. If the real part of $\lambda_i\left(A+\Theta(0)\right)$ is positive, then the TD system be described by (3.1) is unstable.*

*On the other hand, if there exists no real solution $\Theta_j(t)$ for (3.1), we select $\Theta_j(t) = A_j,\ j = 1, 2, \ldots, N$, and the new variable $z(t)$ is defined to be*

$$z(t) = x(t) + \sum_{j=1}^{N} A_j \int_{-jT}^{0} x_t(\tau)d\tau$$

*with it derivative given by*

$$\dot{z}(t) = \left(A + \sum_{j=1}^{N} A_j\right) x(t)$$

*A sufficient condition for the stability of for this type of delay systems is given below.*

***Theorem 3.2*** [20] *Assume that for some symmetric matrices $R_i > 0$ and $R > 0$ there exists a solution, $P > 0$, of the equation*

$$\hat{A}^T P + P\hat{A} + \hat{A}^T P(\sum_{k=1}^{N} A_k R^{-1} A_k^T kT)P\hat{A} = -(R + \sum_{k=1}^{N} R_k kT)$$

*where* $\hat{A} = A + \sum_{j=1}^{N} A_j$

*Then the system (3.1) is asymptotically stable if the system*

$$x(t) + \sum_{j=1}^{N} A_j \int_{-jT}^{0} x_t(\tau)d\tau = 0 \tag{3.18}$$

*is asymptotically stable.*

*A sufficient condition for ensure the stability of (3.18) is given below :*

***Lemma 3.3*** [20] *If $\sum_{j=1}^{N} \|A_j\| jT < 1$, then (3.18) is asymptotically stable.*

## 3.3 Illustrative examples

### Example 3.1

*Consider the delay system given by*

$$\dot{x}(t) = -3x(t) + 0.135x(t-T), \quad \phi(-\tau) = 1$$

*For this example* $A_1 = 0.135$*, and using equation (3.16)*

$$\begin{aligned} 0.135 &= \Theta_1(T) = \exp((-3 + \Theta_1(0)T))\Theta_1(0) \\ \Theta_1(0) &= 0.135 \exp((-3 + \Theta_1(0)T)) \\ T &= \frac{\ln(\frac{\Theta_1(0)}{0.135})}{3 - \Theta_1(0)} \end{aligned}$$

*Since* $T > 0$*, and* $\Theta_1(0) > 0$*, we obtain*

$$0.135 < \Theta_1(0) < 3$$

*For this range of* $\Theta_1(0)$*, it can verified that* $x(t) + \Theta(t) \star x_t = 0$ *is asymptotically stable at* $x = 0$*. Thus, this system is asymptotically stable for all* $T$*.*

**Example 3.2** *Consider the following dynamical system*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + A_1 \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}$$

*First, we consider the term* $A_1$ *is zero matrix. We obtain*

$$x(t) = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} x_0$$

$$\lim_{t \to \infty} \|x(t)\| = 0$$

*Then, it is asymptotically stable in the large. If the term* $A_1 = \begin{bmatrix} 0.2 & 1.5 \\ -1 & 0.2 \end{bmatrix}$*, and by the equation (3.16)*

$$\begin{bmatrix} 0.2 & 1.5 \\ -1 & 0.2 \end{bmatrix} = \exp\left( \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \Theta_1(0) \right) \Theta_1(0)$$

*We can solve by the Newton's method*

$$\Theta_1(0) \simeq \begin{bmatrix} 1.2370 & 0.4161 \\ -0.6132 & 0.8901 \end{bmatrix}$$

*Thus one of the equivalent system is given by*

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0.2370 & 1.4161 \\ -0.6132 & -0.1099 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

*with*

$$\lambda\left(A+\Theta_1(0)\right)=\{0.06355\pm 0.9157i\}$$

*By Theorem 3.1, the system is unstable.*

**Example 3.3** *Consider a system*

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}-1 & -3\\ 2 & -5\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}1.66 & -0.697\\ 0.93 & -0.330\end{bmatrix}\begin{bmatrix}x_1(t-1)\\ x_2(t-1)\end{bmatrix}$$

*with*

$$\begin{bmatrix}x_1(\tau)\\ x_2(\tau)\end{bmatrix}=\begin{bmatrix}1\\ 0.5\end{bmatrix},\quad \tau\in[-1,0]$$

*the corresponding* $\Theta_1(0)$ *is the solution of*

$$\Theta_1(0)=e^{-[A+\Theta_1(0)]}A_1$$

*With the aid of the Newton's method*

$$\Theta_1(0)\simeq\begin{bmatrix}1.99936 & 0.00350\\ 0.00248 & 1.00466\end{bmatrix}$$

*Thus one of the equivalent system is given by*

$$\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{bmatrix}=\begin{bmatrix}0.99936 & -2.99650\\ 2.00248 & -3.99534\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix},\quad \begin{bmatrix}z_1(0)\\ z_2(0)\end{bmatrix}=\begin{bmatrix}2.762\\ 1.3161\end{bmatrix}$$

*with*

$$\lambda\left(A+\Theta_1(0)\right)=\{-1.01188,-1.98496\}$$

*By Theorem 3.1, the system is asymptotically stable.*

# Chapter 4

# Hankel Operators

*Consider stable dynamical system with multiple input delays as given by :*

$$\dot{x}(t) = Ax(t) + B_0 u(t) + \sum_{j=1}^{N} B_j u(t - jT),\ x(-\infty) = 0 \tag{4.1a}$$

$$y(t) = Cx(t) + \sum_{j=1}^{N} D_j u(t - jT),\ t \in (-\infty, \infty) \tag{4.1b}$$

*where the input signal* $u \in \mathcal{L}_-^2$, *and* $A$, $B_0$, $C$, $B_j$, *and* $D_j$ *are constant matrices, for* $j = 1, 2, \cdots, N$. *Since the matrix* $D_0$ *does not affect the value of Hankel norm, we let* $D_0 \equiv 0$ *here. Within this chapter, we will discuss the properties of Hankel operators and its adjoint operators.*

## 4.1 Hankel operator

*Taking the Laplace transformation with* $x(0) = 0$ *on both sides of system (4.1), we obtain*

$$G(s) : \begin{cases} \hat{x}(s) = (sI - A)^{-1}(B_0 + \sum_{j=1}^{N} B_j e^{-sjT})\hat{u}(s) \\ \hat{y}(s) = C\hat{x}(s) + \sum_{j=1}^{N} D_j e^{-sjT}\hat{u}(s) \end{cases}$$

*Therefore the transfer function is given by*

$$G(s) = C(sI - A)^{-1}(B_0 + \sum_{j=1}^{N} B_j e^{-sjT}) + \sum_{j=1}^{N} D_j e^{-sjT} \tag{4.2}$$

*The associated impulse response function $h(t)$ of system (4.1) can be expressed as*

$$h(t) = \sum_{j=0}^{N} Ce^{A(t-jT)} B_j H(t-jT) + \sum_{j=1}^{N} D_j \delta(t-jT) \tag{4.3}$$

*where $H(t)$ and $\delta(t)$ denote the Heviside function and Dirac delta function, respectively. Before constructing the Hankel operator for system (4.1), we need to verify the boundedness of system (4.1). Hence we need the next lemma first.*

*Since the delta function $\delta(t)$ doesn't belong to $\mathcal{L}^1(-\infty,\infty)$, so as $h(t)$; we must clarify the space for $h(t)$.*

***Lemma 4.1*** *Suppose $h(t)$ be the corresponding impulse response function of the system (4.1) which in given by (4.3). Then $h(t) \in \mathcal{P}_1((0,\infty); \mathcal{L}(\mathcal{L}^2_-, \mathcal{L}^2_+))$.*

***Proof.*** *Since*

$$\|h\|_{\mathcal{L}(\mathcal{L}^2_-,\mathcal{L}^2_+)} = \sup_{u \in \mathcal{L}^2_-} \frac{\|h(t)u\|}{\|u\|}$$

*By using (4.3), it follows that*

$$\begin{aligned} \|h(t)u\| &= \|\sum_{j=0}^{N} Ce^{A(t-jT)} B_j H(t-jT)u + \sum_{j=1}^{N} D_j \delta(t-jT)u\| \\ &\leq \sum_{j=0}^{N} \|Ce^{A(t-jT)} B_j H(t-jT)\| \|u\| + \sum_{j=1}^{N} \|D_j \delta(t-jT)\| \|u\| \end{aligned}$$

*Then*

$$\begin{aligned} \|h\|_{\mathcal{P}_1((0,\infty);\mathcal{L}(\mathcal{L}^2_-,\mathcal{L}^2_+))} &= \int_0^\infty \|h(t)\| dt \\ &\leq \int_0^\infty \left( \|\sum_{j=0}^{N} Ce^{A(t-jT)} B_j H(t-jT)\| + \|\sum_{j=1}^{N} D_j \delta(t-jT)\| \right) dt \\ &\leq \|C\| \sum_{j=0}^{N} \int_0^\infty \|e^{A(t-jT)}\| \|B_j\| H(t-jT) dt + \sum_{j=1}^{N} \int_0^\infty \|D_j\| \delta(t-jT) dt \\ &\leq \|C\| \sum_{j=0}^{N} \|B_j\| \int_{-jT}^\infty \|e^{A(t-jT)}\| dt + \sum_{j=1}^{N} \|D_j\| \end{aligned}$$

*Since $A$ is stable, by using Definition 2.1, there exist $M$, $\alpha > 0$ such that*

$$\|e^{At}\| \leq Me^{-\alpha t}$$

*Therefore*

$$\begin{aligned}\|h\|_{\mathcal{P}_1((0,\infty);\mathcal{L}(\mathcal{L}^2_-,\mathcal{L}^2_+))} &\leq \|C\|\sum_{j=0}^{N}\|B_j\|M\int_{jT}^{\infty}e^{-\alpha(t-jT)}dt+\sum_{j=1}^{N}\|D_j\| \\ &= \frac{M}{\alpha}\|C\|\sum_{j=0}^{N}\|B_j\|+\sum_{j=1}^{N}\|D_j\|<\infty\end{aligned}$$

*This concludes our proof.* ■

***Lemma 4.2*** *Suppose the system (4.1) is stable and there are constants $M$ and $\alpha > 0$ such that $\|e^{At}\| \leq Me^{-\alpha t}$. Then the Hankel operator $\Gamma$ of the system (4.1) is bounded, i.e.*

$$\begin{aligned}\|\Gamma\| &\leq \|C\|\sum_{j=0}^{N}\|B_j\|M\int_{jT}^{\infty}e^{-\alpha(t-jT)}dt+\sum_{j=1}^{N}\|D_j\| \\ &= \|C\|\frac{M}{\alpha}\sum_{j=0}^{N}\|B_j\|+\sum_{j=1}^{N}\|D_j\|<\infty\end{aligned}$$

***Proof.*** *From Lemma 4.1, the proof is complete.* ■

***Corollary 4.1*** *Suppose the system (4.1) is stable. Then the output function $y(\cdot)$ belong to $\mathcal{L}^2_+$ whenever the input function $u(\cdot)$ is in $\mathcal{L}^2_-$.*

*The Hankel operator for system (4.1) is defined as*

$$\begin{aligned}\Gamma \; : \; & \mathcal{L}^2_- \to \mathcal{L}^2_+ \\ & u(\cdot) \mapsto y(\cdot) = \Gamma u(\cdot)\end{aligned}$$

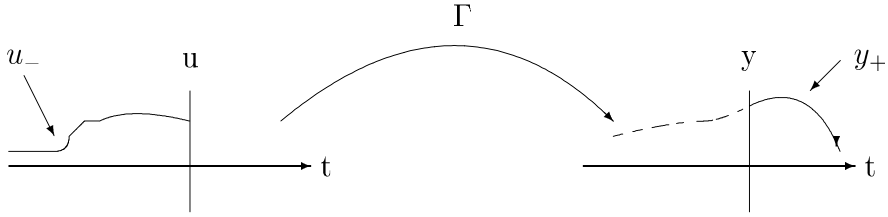



Figure 4.1: The relation between input and output

*Substituting of (4.3) into (2.4) and after some algebraic manipulations, this operator is* *constructed explicitly in the form of*

$$(\Gamma u)(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}B_0u(\tau)d\tau +$$
$$\begin{cases} \sum_{j=1}^{N}\int_{-\infty}^{\min\{jT,t\}} Ce^{A(t-\tau)}B_ju(\tau-jT)d\tau + \sum_{j=i}^{N} D_ju(t-jT), & (i-1)T < t < iT, \\ & i = 1,2,...,N \\ \sum_{j=1}^{N}\int_{-\infty}^{jT} Ce^{A(t-\tau)}B_ju(\tau-jT)d\tau, & t \geq NT \end{cases} \quad (4.4)$$

*due to the fact that $u(t) \equiv 0$ for all positive $t$. The boundedness of $\Gamma$ is obtained directly from Corollary 4.1.*

*The compactness of $\Gamma$ is will discussed in Theorem 4.1. We need the following lemma, first.*

***Lemma 4.3*** *Suppose $D \in \mathbb{R}^{p\times m}$, then $D \neq 0$ if and only if $(D^T)_{k,k} \neq 0$ for all $1 \leq k \leq m$.*

***Proof. (Sufficiency)*** *The matrix $D$ can be expressed as*

$$D = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix}$$

*then it follows $(D^TD)_{k,k} = \sum_{j=1}^{p} d_{jk}^2$. Suppose $(D^TD)_{k,k} = 0$, for all $1 \leq k \leq m$. Then $d_{ij} = 0$, for all $1 \leq i \leq p$, $1 \leq j \leq m$, which means that $D = 0$. It is a contradiction. We have $(D^TD)_{k,k} \neq 0$ for all $1 \leq k \leq m$.*

***(Necessity)*** *The necessity is obviously true.* ■

*When there exist No $D_j$ terms, or equivalently $D_j \equiv 0$, $j = 1,2,\cdots,N$, form Theorem* 8.2.4 *of Curtain and Zwart [5], $\Gamma$ is a compact operator. Here we want to examine the compactness of $\Gamma$ with nonzero $D_j$ terms.*

*Instead of analyzing the case with constant matrices $D_j$, $j = 1,2,\cdots,N$, the compactness of Hankel operator is discussed here with a time-varying matrices $D_j(t)$, $j = 1,2,\cdots,N$.*

***Theorem 4.1*** *The Hankel operator defined in (4.4) is compact if and only if for each $j$, $1 \leq j \leq N$, $D_j(t) = 0$ for almost everywhere $t \in [0, jT]$.*

***Proof.* (Necessity)** *Assume there exist a $j$ and a set $E \subset [0, jT]$ such that $D_j \neq 0$, for all $t \in E$ with the properties:*

1. *The measure of $E$, $\mu(E)$, is positive,*

2. *If $E \cap (a,b) \neq \emptyset$, then $\mu(E \cap (a,b)) > 0$, for any open interval $(a,b)$.*

*Fixed $k$ with $1 \leq k \leq m$, let*

$$\begin{aligned} V &= \{t - jT | t \in E\} \subset [-jT, 0] \\ (u^n)_i(t) &= \begin{cases} \dfrac{\chi_{[b-jT-\frac{\mu(E)}{n}, b-jT] \cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT] \cap V)}}, & i = k \\ 0, & 1 \leq i \neq k \leq m \end{cases} \end{aligned}$$

*for each $n \in \mathbb{N}$, where $(u^n)_i$ denote the $i$-th component of a vector $u^n$ and $\chi$ is a simple function acting on a set $F$. Then it follows*

$$\int_{-\infty}^{0} u^n(t)^T u^n(t) dt = \int_{-\infty}^{0} [(u^n)_k(t)]^2 \, dt = \int_{b-jT-\frac{\mu(E)}{n}}^{b-jT} \left( \frac{\chi_{[b-jT-\frac{\mu(E)}{n}, b-jT] \cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT] \cap V)}} \right)^2 dt = 1.$$

*Hence$\{u^n(t)\}$ is a bounded sequence in $\mathcal{L}^2_-$. Now for any $n$, $\ell \in \mathbb{N}$ and $\ell < n < \infty$,*

$$\begin{aligned} & \|D_j(t)u^n(t-jT) - D_j(t)u^\ell(t-jT)\|^2_{\mathcal{L}^2_+} \\ &= \int_0^\infty [u^n(t-jT) - u^\ell(t-jT)]^T D_j^T(t) D_j(t) [u^n(t-jT) - u^\ell(t-jT)] dt \\ &= \int_{-jT}^{0} (D_j^T(t+jT) D_j(t+jT))_{k,k} \, [(u^n)_k(t) - (u^\ell)_k(t)]^2 dt \end{aligned} \tag{4.5}$$

*Since*

$$\chi_{[b-jT-\frac{\mu(E)}{\ell}, b-jT] \cap V}(t) = \chi_{[b-jT-\frac{\mu(E)}{\ell}, b-jT-\frac{\mu(E)}{n}] \cap V}(t) + \chi_{[b-jT-\frac{\mu(E)}{n}, b-jT] \cap V}(t)$$

*it follows that*

$$
\begin{aligned}
&(u^n)_k(t) - (u^\ell)_k(t) \\
&= \frac{\chi_{[b-jT-\frac{\mu(E)}{n}, b-jT]\cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT]\cap V)}} - \frac{\chi_{[b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}} \\
&= -\frac{\chi_{[b-jT-\frac{\mu(E)}{\ell}, b-jT-\frac{\mu(E)}{n}]\cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}} + \chi_{[b-jT-\frac{\mu(E)}{n}, b-jT]\cap V}(t) \\
&\quad \left(\frac{1}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT]\cap V)}} - \frac{1}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}}\right)
\end{aligned}
\tag{4.6}
$$

*The substitution of (4.6) into (4.5) leads to*

$$
\begin{aligned}
&\|D_j(t)u^n(t-jT) - D_j(t)u^\ell(t-jT)\|^2_{\mathcal{L}^2_+} \\
&= \int_{b-jT-\frac{\mu(E)}{\ell}}^{b-jT-\frac{\mu(E)}{n}} (D_j^T(t+jT)D_j(t+jT))_{k,k} \left(\frac{\chi_{[b-jT-\frac{\mu(E)}{\ell}, b-jT-\frac{\mu(E)}{n}]\cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}}\right)^2 dt \\
&\quad + \int_{b-jT-\frac{\mu(E)}{n}}^{b-jT} (D_j^T(t+jT)D_j(t+jT))_{k,k}\ \chi_{[b-jT-\frac{\mu(E)}{n}, b-jT]\cap V}(t) \\
&\quad \left(\frac{1}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT]\cap V)}} - \frac{1}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}}\right)^2 dt \\
&\geq \int_{b-jT-\frac{\mu(E)}{n}}^{b-jT} (D_j^T(t+jT)D_j(t+jT))_{k,k} \left(1 - \frac{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT]\cap V)}}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}}\right)^2 \\
&\quad \left(\frac{\chi_{[b-jT-\frac{\mu(E)}{n}, b-jT]\cap V}(t)}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT]\cap V)}}\right)^2 dt = s\left(1 - \frac{\sqrt{\mu([b-jT-\frac{\mu(E)}{n}, b-jT]\cap V)}}{\sqrt{\mu([b-jT-\frac{\mu(E)}{\ell}, b-jT]\cap V)}}\right)^2
\end{aligned}
$$

*in which*

$$
s = \int_{b-jT-\frac{\mu(E)}{n}}^{b-jT} (D_j^T(t+jT)D_j(t+jT))_{k,k} dt \neq 0
$$

*Let* $\{u^{n_j}(t)\}$ *be a subsequence of* $\{u^n(t)\}$*. There exists* $N \in \mathbb{N}$, $n_k$, $n_\ell > N$, $n_k > n_\ell$, *and* $n_k \to \infty$ *as* $k \to \infty$, *we obtain*

$$
\|D_j(t)u^{n_k}(t-jT) - D_j(t)u^{n_\ell}(t-jT)\|^2_{\mathcal{L}^2_+} \geq s\left(1 - \frac{\sqrt{\mu([b-jT-\frac{\mu(E)}{n_k}, b-jT]\cap V)}}{\sqrt{\mu([b-jT-\frac{\mu(E)}{n_\ell}, b-jT]\cap V)}}\right)^2
$$

*or equivalently,*

$$\lim_{k\to\infty} \|D_j(t)u^{n_k}(t-jT) - D_j(t)u^{n_\ell}(t-jT)\|^2_{\mathcal{L}^2_+} \geq s$$

*When $k$ is ranged from 1 to $N$, the above relation holds for all $k$. By the definition of Cauchy sequence, the sequence $\{D_j(t)u^n(t-jT)\}$ does not contain a Cauchy subsequence in $\mathcal{L}^2_+$. This means that it does not contain a convergent subsequence in $\mathcal{L}^2_+$. Therefore, $\Gamma$ is not compact if there exist a $j$ and a set $E \subset [0, jT]$ such that $D_j(t) \neq 0$ for all $t \in E$.*
***(Sufficiency)*** *Suppose $D_j(t) = 0$ for almost everywhere $t \in [0, jT]$ with $j = 1, 2, \cdots, N$, and let $\{u^n(t)\}$ be a bounded sequence in $\mathcal{L}^2_-$, then*

$$\begin{aligned}&\int_0^\infty [u^n(t-jT) - u^m(t-jT)]^T D_j(t)^T D_j(t)[u^n(t-jT) - u^m(t-jT)]dt \\ = &\int_{-jT}^0 [u^n(t) - u^m(t)]^T D_j(t+jT)^T D_j(t+jT)[u^n(t) - u^m(t)]dt = 0\end{aligned}$$

*for all $n < \infty$ and $n, m \in \mathbb{N}$. Hence the sequence $\{D_j(t)u^n(t-jT)\}$ is a Cauchy sequence in $\mathcal{L}^2_+$ and then it is convergent in $\mathcal{L}^2_+$. Therefore, $\Gamma$ is compact.* ■

## 4.2 Adjoint Hankel operator

*It is noted that $(\Gamma u)(t)$ as given in (4.4) is the sum of three types of operators, $\Gamma_0$, $\Gamma_j$, and $\Gamma_{d_j}$, $j = 1, 2, .., N$, to be defined, respectively, as*

$$\begin{aligned}(\Gamma_0 u)(t) &= \int_{-\infty}^0 Ce^{A(t-\tau)}B_0 u(\tau)d\tau \\ (\Gamma_j u)(t) &= \int_{-\infty}^{\min\{t,jT\}} Ce^{A(t-\tau)}B_j u(\tau - jT)d\tau \\ &= \int_{-\infty}^{\min\{0,t-jT\}} Ce^{A(t-\tau)}\tilde{B}_j u(\tau)d\tau \\ (\Gamma_{d_j} u)(t) &= \begin{cases} D_j u(t-jT), & 0 < t < jT \\ 0, & t \geq jT \end{cases}\end{aligned}$$

*where $\tilde{B}_j = e^{-AjT}B_j$. The derivation of the adjoint of $\Gamma$ is easier if one computes the adjoints for $\Gamma_0$, $\Gamma_j$, and $\Gamma_{d_j}$, separately. Since the adjoint operator $\Gamma_j^*$ can be determined according to*

$$\langle v, \Gamma_j u\rangle_{\mathcal{L}^2_+} = \langle \Gamma_j^* v, u\rangle_{\mathcal{L}^2_-}$$

*Then,*

$$\begin{aligned}
\langle v, \Gamma_j u\rangle_{\mathcal{L}^2_+} &= \langle \Gamma_j^* v, u\rangle_{\mathcal{L}^2_-} \\
&= \int_0^\infty v^T(\tau) \int_{-\infty}^{\min\{t,jT\}} Ce^{A(\tau-t)} B_j u(t-jT) dt d\tau \\
&= \int_0^\infty v^T(\tau) \int_{-\infty}^{\min\{0,t-jT\}} Ce^{A(\tau-t-jT)} B_j u(t) dt d\tau \\
&= \int_{-\infty}^0 \left( \int_{\max\{0,t+jT\}}^\infty v^T(\tau) Ce^{A(\tau-t)} \tilde{B}_j d\tau \right) u(t) dt \\
&= \int_{-\infty}^0 \left( \int_{\max\{0,t+jT\}}^\infty \tilde{B}_j^T e^{A(\tau-t)} C^T v^(\tau) d\tau \right)^T u(t) dt
\end{aligned}$$

*We obtain*

$$(\Gamma_j^* v)(t) = \int_{\max\{0,t+jT\}}^\infty \tilde{B}_j^T e^{A^T(\tau-t)} C^T v(\tau) d\tau$$

*Similarly, the adjoints of* $\Gamma_0$ *and* $\Gamma_{d_j}$ *are found to be*

$$\begin{aligned}
(\Gamma_0^* v)(t) &= \int_0^\infty B_0^T e^{A^T(\tau-t)} C^T v(\tau) d\tau \\
(\Gamma_{d_j}^* v)(t) &= \begin{cases} D_j^T v(t+jT), & -jT < t < 0 \\ 0, & t \le -jT \end{cases}
\end{aligned}$$

*Thus the summation of the above three adjoints gives us the adjoint operator* $\Gamma^*$ *as*

$$\Gamma^* \ : \ \mathcal{L}^2(0,\infty;\mathbb{R}^p) \longrightarrow \mathcal{L}^2(-\infty,0;\mathbb{R}^m) \ : \ v(\cdot) \mapsto (\Gamma^* v)(\cdot)$$

*where*

$$(\Gamma^* v)(t) = \int_0^\infty B_0^T e^{A^T(\tau-t)} C^T v(\tau) d\tau +$$
$$\begin{cases} \sum_{j=1}^N \int_{\max\{0,t+jT\}}^\infty \tilde{B}_j^T e^{A^T(\tau-t)} C^T v(\tau) d\tau + \sum_{j=i}^N D_j^T v(t+jT), \\ \qquad -iT < t < -(i-1)T, \ i = 1,2,..,N \\ \sum_{j=1}^N \int_0^\infty \tilde{B}_j^T e^{A^T(\tau-t)} C^T v(\tau) d\tau, \quad t \le -NT \end{cases} \tag{4.7}$$

***Corollary 4.2*** $\Gamma^*$ *is not compact operator if and only if for each* $j$, $1 \le j \le N$, $D_j(t) = 0$ *for almost everywhere* $t \in [-jT, 0]$.

***Proof.*** *The proof of corollary 4.2 is similar to Theorem 4.1's.* ■

## 4.3 Essential spectrum

***Lemma 4.4*** *The Hankel operator* $\Gamma$ *is a Fredholm operator.*

***Proof.*** *This follows directly from the definition of Fredholm operator.* ■

***Definition 4.1*** The essential spectrum of $\Gamma$, $\rho_{ess}(\Gamma)$, is defined to be

$$\rho_{ess}(\Gamma)=\{\lambda|\lambda \in \mathbb{C} \text{ such that } \lambda I - \Gamma \text{ is not a Fredholm operator}\}$$

with the spectrum radius defined by

$$\sigma_{ess}(\Gamma) = \sup\{|\lambda||\lambda \in \rho_{ess}(\Gamma)\}$$

By this definition, it follows that the spectrum radius $\sigma_{ess}(\Gamma)$ which is equal to

$$\sigma_{ess}(\Gamma) = \inf\{\|\Gamma - K\||K \text{ is compact}\}$$

*From Theorem 4.1 when the constant matrix* $D_j \not\equiv 0$*, the Hankel operator* $\Gamma$ *for the system (4.1) is noncompact. Pandolfi[27] computes the essential spectrum of* $\Gamma^*\Gamma$ *for the condition* $T = 1$*. The extension of Pandolfi's result to general* $T$ *is given by the following lemma:*

***Lemma 4.5*** *The essential spectrum* $\Gamma^*\Gamma$ *is a finite set with its elements are identified by the algebraic equation*

$$\det(\lambda I - \mathcal{D}^T\mathcal{D}) = 0, \qquad \mathcal{D} = \begin{bmatrix} D_1 & \cdots & D_N \\ \vdots & & \vdots \\ D_N & \cdots & 0 \end{bmatrix}$$

*where* $I$ *stands for identity matrix with appropriate dimension.*

***Proof.*** *We can partition the operator* $\Gamma$ *into two parts:*

$$\Gamma = \Gamma_c \oplus \Gamma_d \tag{4.8}$$

*where $\Gamma_c$ is compact but $\Gamma_d$ is noncompact. Here $\Gamma_d$ and its adjoint are given by*

$$\Gamma_d u(t) = \begin{cases} D_1 u(t-T) + D_2 u(t-2T) + \cdots + D_N u(t-NT), & 0 < t < T \\ D_2 u(t-2T) + D_3 u(t-3T) + \cdots + D_N u(t-NT), & T < t < 2T \\ \vdots \\ D_N u(t-NT), & (N-1)T < t < NT \end{cases} \tag{4.9a}$$

$$\Gamma_d^* v(t) = \begin{cases} D_1^T v(t+T) + D_2^T v(t+2T) + \cdots + D_N^T v(t+NT), & -T < t < 0 \\ D_2^T v(t+2T) + D_3^T v(t+3T) + \cdots + D_N^T v(t+NT), & -2T < t < -T \\ \vdots \\ D_N^T v(t+NT), & -NT < t < -(N-1)T \end{cases} \tag{4.9b}$$

*Follows the results from Zames and Mitter [32] that* "If $X$, $Y$ are any pair of operators in a Hilbert space, then $X \sim Y$ means that $X - Y$ is compact. The symbol $\sim$ denotes equivalence modulo the compact operators (i.e., in a Calkin Algebra ). It follows from the definition of essential spectrum that if $X \sim Y$, then $X$, $Y$ have identical essential spectra.".
*By (4.8),*

$$\Gamma^*\Gamma = (\Gamma_c \oplus \Gamma_d)^*(\Gamma_c \oplus \Gamma_d) = \Gamma_c^*\Gamma_c \oplus \Gamma_c^*\Gamma_d \oplus \Gamma_d^*\Gamma_c \oplus \Gamma_d^*\Gamma_d$$

*and using property that the composition of a compact operator is also compact and so is its adjoint. Thus, the noncompactness of $\Gamma^*\Gamma$ is due to the term $\Gamma_d^*\Gamma_d$. That is, the essential spectrum of $\Gamma^*\Gamma$ coincides with the spectrum of the noncompact part $\Gamma_d^*\Gamma_d$.*
*By viewing (4.9), we decompose the signal $u(t)$ in the following way :*

$$u_j(t) = u(t - jT), \; j = 1, 2, \cdots, N$$

*and then the $\Gamma_d u(\cdot)$ can be rewritten in terms of $u_j(t)$, i.e.*

$$\begin{bmatrix} \Gamma_d u(t)\chi_{[0,T]}(t) \\ \Gamma_d u(t)\chi_{[T,2T]}(t) \\ \vdots \\ \Gamma_d u(t)\chi_{[(N-1)T,NT]}(t) \end{bmatrix} = \begin{bmatrix} D_1 & D_2 & \cdots & D_{N-1} & D_N \\ D_2 & D_3 & \cdots & D_N & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_N & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N-1}(t) \\ u_N(t) \end{bmatrix}$$

*Thus, the essential spectrum* $\Gamma_d^*\Gamma_d$ *is a finite set with its elements are identified by the algebraic equation*

$$\det(\lambda I - \mathcal{D}^T\mathcal{D}) = 0, \qquad \mathcal{D} = \begin{bmatrix} D_1 & \cdots & D_N \\ \vdots & & \vdots \\ D_N & \cdots & 0 \end{bmatrix} \tag{4.10}$$

*where I stands for identity matrix with appropriate dimension. Hence we prove Lemma 4.5.*
■

**Example 4.1** Let $G(s) = \frac{1}{s+1} + e^{-s}$, and the state-space realization of $G(s)$ is given by

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) \\ y(t) &= x(t) + u(t-1) \end{aligned}$$

Since $D_1 = 1$, by Lemma 4.5, then the essential spectrum of Hankel operator is 1.

**Example 4.2** Given the system

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) + u(t-1) + u(t-2) \\ y(t) &= x(t) + u(t-1) + u(t-2) \end{aligned}$$

Since $D_1 = 1, D_2 = 1$, by Lemma 4.5,

$$\det\left(\lambda I - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0$$

Then the essential spectrum of Hankel operator is 2.618.

Once the essential spectrum of $\Gamma$ is computed, the Hankel norm or the maximal singular value must be greater than $\sqrt{\sigma_{ess}(\Gamma^*\Gamma)}$.

## 4.4 Input/Output maps

In this section, we want construct input/output map for $\Gamma$.

Since $y(t) = Cx(t) + \sum_{j=1}^{N} D_j u(t - jT),\ t > 0$, only the input signal $u(t)$ in $[-NT, 0]$ will directly affect the output signal $y(t)$. For each signal in $[-NT, 0] \subset \mathcal{L}_-^2$, we can divide it into $N$ segments, and each segment is shift in time such that its domain become $[-T, 0]$. These processes are shown in Figure 4.2. After stacking these segments, we arrive a new space $[-T, 0]^N$. Thus, any input signal in $\mathcal{L}^2_-$ can be partitioned as follows

$$\begin{aligned} u_{1j}(t) &= u(t-(j-1)T), \quad -T \le t \le 0,\ 1 \le j \le N \\ u_2(t) &= u(t), \qquad\qquad\qquad t < -NT \end{aligned}$$

and let $\mathbf{u}_1(t) = [u_{11}(t), \cdots, u_{1N}(t)]^T$. Then

$$\begin{pmatrix} \mathbf{u}_1(t) \\ u_2(t) \end{pmatrix} \subset \mathcal{U}$$

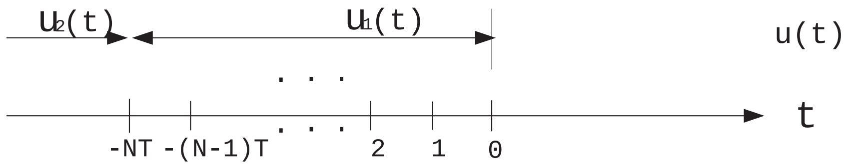
where $\mathcal{U} = \mathcal{L}^2[-T, 0]^N \times \mathcal{L}^2(-\infty, -NT]$.

Similarly, the output signal in $\mathcal{L}^2_+$ can be rewritten as a signal belongs to $\mathcal{Y}$, i.e.

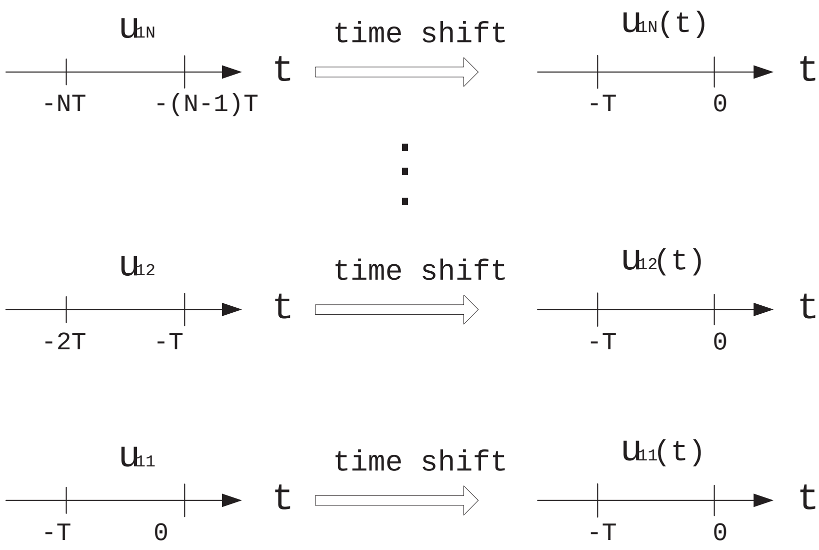
$$\begin{aligned} y_{1j}(t) &= y(t+jT), \qquad -T \le t \le 0,\ 1 \le j \le N \\ y_2(t) &= y(t), \qquad\qquad NT > t \end{aligned}$$

and let $\mathbf{y}_1(t) = [y_{11}(t), \cdots, y_{1N}(t)]^T$. Then

$$\begin{pmatrix} \mathbf{y}_1(t) \\ y_2(t) \end{pmatrix} \subset \mathcal{Y}$$




$(a)$ original signal in $[-NT, 0]$




$(b)$ new signal in $[-T, 0]^N$

Figure 4.2: Stacking the input signal from $[-NT, 0]$ to $[-T, 0]^N$

We recognize this $\mathcal{U}$ and $\mathcal{Y}$ as our new input and output space. Thus we define the following spaces.

**Definition 4.2** Define the "state space" $\mathcal{X} = \mathbb{R}^n \times \mathcal{L}^2[-T,0]^N$, the signal space of the past input $\mathcal{U} = \mathcal{L}^2(-\infty,-NT] \times \mathcal{L}^2[-T,0]^N$, and the signal space of the future output $\mathcal{Y} = \mathcal{L}^2[-T,0]^N \times \mathcal{L}^2[NT,\infty)$.

Let $x_N$ denote the value of state at $t = NT$, i.e. $x_N = x(NT)$.

**Definition 4.3** The input map $\Psi : \mathcal{U} \to \mathcal{X}$ is defined by

$$\Psi \begin{pmatrix} \mathbf{u}_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} x_N \\ \mathbf{u}_1 \end{bmatrix}$$

We can realize $\Psi$ into four parts :

$$\Psi \triangleq \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$$

i.e.

$$\begin{aligned} (\Psi_{11}\mathbf{u}_1)(t) + (\Psi_{12}u_2)(t) &= x_N \\ (\Psi_{21}\mathbf{u}_1)(t) + (\Psi_{22}u_2)(t) &= \mathbf{u}_1 \end{aligned}$$

which implies

$$\begin{aligned} (\Psi_{11}\mathbf{u}_1)(t) &= \int_{-T}^{0} e^{-A\tau}\mathcal{A}\mathcal{B}(A)\mathbf{u}_1(\tau)d\tau \\ (\Psi_{12}u_2)(t) &= \int_{-\infty}^{-NT} e^{-A\tau}\mathcal{B}(A)u_2(\tau)d\tau \\ (\Psi_{21}\mathbf{u}_1)(t) &= \mathbf{u}_1 \\ (\Psi_{22}u_2)(t) &= 0 \end{aligned}$$

with

$$\begin{aligned} \mathcal{B}(A) &= \sum_{j=0}^{N} e^{A(N-j)T}B_j \\ \mathcal{A} &= \left[\ I, e^{AT}, e^{2AT}, \cdots e^{A(N-1)T}\ \right] \end{aligned}$$

**Definition 4.4** The output map $\Phi : \mathcal{X} \to \mathcal{Y}$ is defined by

$$\Phi \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ y_2 \end{bmatrix}$$

We can realize $\Phi$ into four parts :

$$\Phi \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

i.e.

$$(\Phi_{11}x_N)(t) + (\Phi_{12}\mathbf{u}_1)(t) = \mathbf{y}_1$$
$$(\Phi_{21}x_N)(t) + (\Phi_{22}\mathbf{u}_1)(t) = y_2$$

which implies

$$\begin{aligned}
(\Phi_{11}x_N)(t) &= Ce^{At}\widehat{\mathcal{A}}x_N \\
(\Phi_{12}\mathbf{u}_1)(t) &= -\int_t^0 Ce^{-A(\tau-t)}\mathcal{B}\mathbf{u}_1(\tau)d\tau - \int_{-T}^0 Ce^{-A(\tau-t)}\widehat{\mathcal{B}}(A)\mathbf{u}_1(\tau)d\tau + \mathcal{D}\mathbf{u}_1(t) \\
(\Phi_{21}x_N)(t) &= Ce^{-A(NT-t)}x_N \\
(\Phi_{22}\mathbf{u}_1)(t) &= 0
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\mathcal{A}} &= \begin{bmatrix} e^{-A(N-1)T}, e^{-A(N-2)T}, e^{-A(N-3)T} \cdots, I \end{bmatrix}^T \\
\mathcal{B} &= \begin{bmatrix} B_1 & \cdots & B_N \\ \vdots & & \vdots \\ B_N & \cdots & \mathbf{0} \end{bmatrix} \\
\widehat{\mathcal{B}}(A) &= \begin{bmatrix} \sum_{j=2}^{N} e^{-A(j-1)T}B_j & \cdots & \cdots & e^{-NT}B_N & \mathbf{0} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ e^{-AT}B_N & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{bmatrix}
\end{aligned}$$

and $\mathcal{D}$ is defined as before.

**Lemma 4.6** *The adjoint map $\Psi^*$ of the input map $\Psi$ is described as follow:*

$$\Psi^* \begin{pmatrix} x_N \\ \mathbf{u}_1(t) \end{pmatrix} = \begin{bmatrix} \mathcal{B}(A)^T \mathcal{A} e^{-A^T t} x_N + \mathbf{u}_1(t) \\ \mathcal{B}(A)^T e^{-A^T t} x_N \end{bmatrix}$$

**Proof.** The adjoint operator of $\Psi$ is defined by

$$< \begin{pmatrix} \mathbf{u}_1 \\ u_2 \end{pmatrix}, \Psi^* \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} >_{\mathcal{U}} = < \Psi \begin{pmatrix} \mathbf{u}_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} >_{\mathcal{X}}$$

Since

$$\begin{aligned}
& < \Psi \begin{pmatrix} \mathbf{u}_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} >_{\mathcal{X}} \\
&= < \Psi_{11}\mathbf{u}_1 + \Psi_{12}u_2, x_N >_{\mathbb{R}^N} + < \Psi_{21}\mathbf{u}_1 + \Psi_{22}u_2, \mathbf{u}_1 >_{\mathcal{L}^2(-T,0)^N} \\
&= (\Psi_{11}\mathbf{u}_1)^T x_N + (\Psi_{12}u_2)^T x_N + \int_{-T}^{0} \mathbf{u}_1(\tau)^T \mathbf{u}_1(\tau) d\tau \\
&= \int_{-T}^{0} \mathbf{u}_1(\tau)^T \left( \mathcal{B}(A)^T \mathcal{A}^T e^{-A^T\tau} x_N + \mathbf{u}_1(\tau) \right) d\tau + \int_{-\infty}^{-NT} u_2(\tau)^T \mathcal{B}(A)^T e^{-A^T\tau} x_N d\tau \\
&= < \begin{pmatrix} \mathbf{u}_1 \\ u_2 \end{pmatrix}, \Psi^* \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} >_{\mathcal{U}}
\end{aligned}$$

Therefore,

$$\Psi^* \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} = \begin{pmatrix} \Psi_{11}^* & \Psi_{21}^* \\ \Psi_{12}^* & \Psi_{22}^* \end{pmatrix} \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix} = \begin{bmatrix} \mathcal{B}(A)^T \mathcal{A}^T e^{-A^T t} x_N + \mathbf{u}_1(t) \\ \mathcal{B}(A)^T e^{-A^T t} x_N \end{bmatrix}$$

■

**Lemma 4.7** *The adjoint map $\Phi^*$ of the output map $\Phi$ is described as follow:*

$$\Phi^* \begin{pmatrix} \mathbf{y}_1(t) \\ y_2(t) \end{pmatrix} = \begin{bmatrix} \int_{-T}^{0} \widehat{\mathcal{A}}^T e^{A^T\tau} C^T \mathbf{y}_1(\tau) d\tau + \int_{NT}^{\infty} e^{-A^T(NT-\tau)} C^T y_2(\tau) d\tau \\ \int_{t}^{-T} \mathcal{B}(A)^T e^{-A^T(t-\tau)} C^T \mathbf{y}_1(\tau) d\tau - \int_{-T}^{0} \widehat{\mathcal{B}}(A)^T e^{-A^T(t-\tau)} C^T \mathbf{y}_1(\tau) d\tau + \mathcal{D}^T \mathbf{y}_1(t) \end{bmatrix}$$

**Proof.** The adjoint operator of $\Phi$ is defined by

$$< \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix}, \Phi^* \begin{pmatrix} \mathbf{y}_1 \\ y_2 \end{pmatrix} >_{\mathcal{X}} = < \Phi \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{y}_1 \\ y_2 \end{pmatrix} >_{\mathcal{Y}}$$

Since

$$\begin{aligned}
& < \Phi \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{y}_1 \\ y_2 \end{pmatrix} >_{\mathcal{Y}} \\
&= < \Phi_{11}x_N + \Phi_{12}\mathbf{u}_1, \mathbf{y}_1 >_{\mathcal{L}^2(-T,0)^N} + < \Phi_{21}x_N + \Phi_{22}\mathbf{u}_1, y_2 >_{\mathcal{L}^2(NT,\infty)} \\
&= x_N^T \left( \int_{-T}^{0} \widehat{\mathcal{A}}^T e^{A^T\tau} C^T \mathbf{y}_1(\tau) d\tau + \int_{NT}^{\infty} e^{-A^T(NT-\tau)} C^T y_2(\tau) d\tau \right) \\
&- \int_{-T}^{0} \mathbf{u}_1(t) \left( \int_{-T}^{t} \mathcal{B}(A)^T e^{-A^T(t-\tau)} C^T \mathbf{y}_1(\tau) d\tau + \int_{-T}^{0} \widehat{\mathcal{B}}(A)^T e^{-A^T(t-\tau)} C^T \mathbf{y}_1(\tau) d\tau + \mathcal{D}^T \mathbf{y}_1(t) \right) dt
\end{aligned}$$

Therefore,

$$\Phi^* \begin{pmatrix} \mathbf{y}_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \int_{-T}^{0} \widehat{\mathcal{A}}^T e^{A^T \tau} C^T \mathbf{y}_1(\tau) d\tau + \int_{NT}^{\infty} e^{-A^T(NT-\tau)} C^T y_2(\tau) d\tau \\ \int_{t}^{-T} \mathcal{B}(A)^T e^{-A^T(t-\tau)} C^T \mathbf{y}_1(\tau) d\tau - \int_{-T}^{0} \widehat{\mathcal{B}}(A)^T e^{-A^T(t-\tau)} C^T \mathbf{y}_1(\tau) d\tau + \mathcal{D}^T \mathbf{y}_1(t) \end{bmatrix}$$

■

The product of $\Psi$ and $\Phi$ is given by

$$\left( \Phi\Psi \begin{pmatrix} \mathbf{u}_1 \\ u_2 \end{pmatrix} \right) = \Phi \begin{pmatrix} x_N \\ \mathbf{u}_1 \end{pmatrix}$$

$$= \begin{bmatrix} Ce^{At}\widehat{\mathcal{A}}x_N - \int_t^0 Ce^{-A(\tau-t)}\mathcal{B}\mathbf{u}_1(\tau)d\tau - \int_{-T}^0 Ce^{-A(t-\tau)}\widehat{\mathcal{B}}(A)\mathbf{u}_1(\tau)d\tau + \mathcal{D}\mathbf{u}_1(t) \\ Ce^{-A(NT-t)}x_N \end{bmatrix}$$

Define $\hat{\Gamma}$ as the representation of $\Gamma$ in terms of $\mathcal{U}$ and $\mathcal{Y}$, i.e. $\Gamma \ : \mathcal{L}_-^2 \to \mathcal{L}_+^2$ is equivalent to $\hat{\Gamma} \ : \mathcal{U} \to \mathcal{Y}$. After some algebraic operator, we arrive at $\hat{\Gamma} = \Phi\Psi$, hence $\hat{\Gamma}^* = \Psi^*\Phi^*$. The controllable and observable gramians are $\mathcal{P} = \Psi\Psi^*$ and $\mathcal{Q} = \Phi^*\Phi$, respectively, i.e.

$$\mathcal{P} \begin{pmatrix} x_N \\ \mathbf{u}_1(t) \end{pmatrix} = (\Psi\Psi^*) \begin{pmatrix} x_N \\ \mathbf{u}_1(t) \end{pmatrix} = \begin{bmatrix} P & \mathcal{W}_1 \\ \mathcal{W}_1^* & I \end{bmatrix} \begin{bmatrix} x_N \\ \mathbf{u}_1(t) \end{bmatrix}$$

where

$$\begin{aligned} P &= \int_{-\infty}^0 e^{-A\tau}\mathcal{B}(A)\mathcal{B}(A)^T e^{-A^T\tau} d\tau \\ \mathcal{W}_1\mathbf{u}_1 &= \int_{-T}^0 \mathcal{A}e^{-At}\mathcal{B}(A)\mathbf{u}_1(t)dt \\ \mathcal{W}_1^* x_N &= \mathcal{B}(A)^T e^{-A^T t}\mathcal{A}^T x_N \end{aligned}$$

and

$$Q \begin{pmatrix} x_N \\ \mathbf{u}_1(t) \end{pmatrix} = (\Phi^*\Phi) \begin{pmatrix} x_N \\ \mathbf{u}_1(t) \end{pmatrix} = \left( \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathcal{V} - \mathcal{V}_3 \end{bmatrix} \begin{bmatrix} \widehat{Q} & \mathcal{W}_2 \\ \widehat{\mathcal{A}} & I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathcal{V}_1 - \mathcal{V}_2 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} \mathbf{0} & \mathcal{V}_4 \\ \mathcal{V}_4^* & \mathcal{D}^T Ce^{At}(\mathcal{V}_1 + \mathcal{V}_2) + \mathcal{D}^T \end{bmatrix} \right) \begin{bmatrix} x_N \\ \mathbf{u}_1(t) \end{bmatrix}$$

where

$$
\begin{aligned}
\widehat{Q} &= \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau \\
\mathcal{W}_2\omega &= \int_{-T}^0 \widehat{\mathcal{A}}^T e^{A\tau} C^T C e^{A\tau} \omega(\tau) d\tau \\
\mathcal{V}\omega(t) &= \int_t^{-T} \mathcal{B}(A)^T e^{-A^T(t-\tau)} C^T C e^{A\tau} \omega(\tau) d\tau \\
\mathcal{V}_1 \mathbf{u}_1(\tau) &= \int_0^\tau e^{-At} \mathcal{B}(A) \mathbf{u}_1(t) dt \\
\mathcal{V}_2 \mathbf{u}_1 &= \int_{-T}^0 e^{-At} \widehat{\mathcal{B}}(A) \mathbf{u}_1(t) dt \\
\mathcal{V}_3\omega &= \int_{-T}^0 \widehat{\mathcal{B}}(A)^T e^{-A^T(t-\tau)} C^T C e^{A\tau} \omega(\tau) d\tau \\
\mathcal{V}_4 \mathbf{u}_1 &= \int_{-T}^0 \widehat{\mathcal{A}}^T e^{A^T\tau} C^T \mathcal{D} \mathbf{u}_1(t) dt \\
\mathcal{V}_4^* x_N &= \mathcal{D}^T C e^{At} x_N \widehat{\mathcal{A}}
\end{aligned}
$$

The controllable and observable gramians can be used to compute the singular value of $\hat{\Gamma}$. Let $\sigma$ be the singular value of $\hat{\Gamma}$ and $(u, v)$ be the Schimdt pair of $\hat{\Gamma}$, then we have

$$
\begin{aligned}
\Phi \mathcal{P} \Phi^* v &= \sigma^2 v \\
\Psi^* \mathcal{Q} \Psi u &= \sigma^2 u
\end{aligned}
$$

# Chapter 5

# Hankel Norm Computation

In this chapter, we will compute Hankel norm of the following stable system :

$$\dot{x}(t) = Ax(t) + B_0 u(t) + \sum_{j=1}^{N} B_j u(t - jT), \; x(-\infty) = 0 \tag{5.1a}$$

$$y(t) = Cx(t) + \sum_{j=1}^{N} D_j u(t - jT), \; t \in (-\infty, \infty) \tag{5.1b}$$

The Hankel operator for system (5.1) and its adjoint have been discussed in chapter 4. The Hankel norm of system (5.1) is defined as the operator norm of $\Gamma$, i.e.

**Definition 5.1** The *Hankel norm* of the linear system (5.1) is defined as

$$\|G\|_H = \|\Gamma\| \triangleq \sup_{u \in \mathcal{L}_-^2, u \neq 0} \frac{\|y\|_{\mathcal{L}_+^2}}{\|u\|_{\mathcal{L}_-^2}} = \sup_{x_0 \in \mathbb{R}^n} \sup_{u \in U_0} \frac{\|y\|_{\mathcal{L}_+^2}}{\|u\|_{\mathcal{L}_-^2}}$$

where $y = \Gamma u$, $U_0 = \left\{ u \in \mathcal{L}_-^2, u \neq 0 \,\middle|\, x_0 = \int_{-\infty}^{0} e^{-A\tau} \sum_{j=0}^{N} B_j u(\tau - jT) d\tau, \; x_0 \in \mathbb{R}^n \right\}$.

The norm of $\Gamma$ can be related to the singular value of $\Gamma$ which is defined by :

**Definition 5.2** Hankel *singular vector* $(u, v)$ and *value* $\sigma$ of the operator $\Gamma$ are defined as the nonzero solutions of the following equations

$$\Gamma u = \sigma v, \qquad \Gamma^* v = \sigma u \tag{5.2}$$

For any inner product spaces $U_1$ and $U_2$, we have the following theorem :

**Theorem 5.1** [18] *Let $U_1$, $U_2$ be inner product spaces, and let $T : U_1 \to U_2$ be a linear transformation. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be its singular values. Then*

$$\sigma_k = \min_{codim\ M=k-1} \max_{x\in M} \frac{\|Tx\|}{\|x\|}$$

Since $\mathcal{L}^2_-$, $\mathcal{L}^2_+$ are inner product spaces, by Theorem 5.1, the following result is obvious true.

**Corollary 5.1** *$\mathcal{L}^2_-$, $\mathcal{L}^2_+$ are inner product spaces, $\Gamma : \mathcal{L}^2_- \to \mathcal{L}^2_+$ is a linear transformation. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be its singular values. Then*

$$\sigma_1 = \sup_{u\in\mathcal{L}^2_-} \frac{\|\Gamma u\|}{\|u\|}$$

From corollary (5.1), we know that its norm will be equal to the largest singular value. Therefore, we present two different ways to compute the Hankel norm of the system : one is via variational principle, the other is via Hankel singular value computation.

## 5.1 Variational principle

Fixed $x_0$, since the Hankel operator $\Gamma$ of the system (5.1) is bounded, there exists $\gamma$ such that

$$\sup_{u\in U_0} \frac{\|y\|_{\mathcal{L}^2_+}}{\|u\|_{\mathcal{L}^2_-}} \leq \gamma, \quad \text{or equivalently,} \quad \max_{u\in U_0}\left(\gamma^2\|u\|^2_{\mathcal{L}^2_-} - \|y\|^2_{\mathcal{L}^2_+}\right) \geq 0$$

which is transformed into the optimization problem: Given the cost function defined by

$$\begin{aligned} J(u,x;\gamma) &\triangleq \frac{\gamma^2}{2}\int_{-\infty}^{0} u(t)^{\mathrm{T}}u(t)dt - \frac{1}{2}\int_0^{\infty} y(t)^{\mathrm{T}}y(t)dt \\ &= \frac{\gamma^2}{2}\int_{-\infty}^{0} u(t)^{\mathrm{T}}u(t)dt - \frac{1}{2}\int_{NT}^{\infty} x(t)^{\mathrm{T}}C^{\mathrm{T}}Cx(t)dt \\ &\quad -\frac{1}{2}\sum_{i=0}^{N-1}\int_{iT}^{(i+1)T}\left(Cx(t) + \sum_{j=i+1}^{N} D_j u(t-jT)\right)^{\mathrm{T}}\left(Cx(t) + \sum_{k=i+1}^{N} D_k u(t-kT)\right)dt \end{aligned} \tag{5.3}$$

we want to find $u$ such that $J(u,x;\gamma)$ is minimized subjected to the constraint (5.1a) and choose the smallest possible $\gamma$ such that the optimal cost equals to 0. Using the method of

Lagrange multiplier, the augmented cost function is obtained as

$$\begin{aligned} J_a(x,u,\lambda;\gamma) &= \int_{-\infty}^{0} \left\{ \frac{\gamma^2}{2} u(t)^{\mathrm{T}} u(t) + \lambda(t)^{\mathrm{T}} \left( Ax(t) + B_0 u(t) + \sum_{j=1}^{N} B_j u(t-jT) - \dot{x}(t) \right) \right\} dt \\ &+ \sum_{i=0}^{N-1} \int_{iT}^{(i+1)T} \left\{ -\frac{1}{2} \left( Cx(t) + \sum_{j=i+1}^{N} D_j u(t-jT) \right)^{\mathrm{T}} \left( Cx(t) + \sum_{k=i+1}^{N} D_k u(t-kT) \right) \right. \\ &\left. + \lambda(t)^{\mathrm{T}} \left( Ax(t) + \sum_{j=i+1}^{N} B_j u(t-jT) - \dot{x}(t) \right) \right\} dt \\ &+ \int_{NT}^{\infty} \left\{ -\frac{1}{2} x(t)^{\mathrm{T}} C^{\mathrm{T}} C x(t) + \lambda(t)^{\mathrm{T}} \left( Ax(t) - \dot{x}(t) \right) \right\} dt \end{aligned}$$

By Theorem 2.5 the optimal state $x$, control input $u$, and costate $\lambda$, which minimize $J_a(x,u,\lambda;\gamma)$, must be the solutions of the following equations : for $i = 0, 1, \cdots, N-1$,

$$0 = \begin{cases} \gamma^2 u(t) + \sum_{j=0}^{N} B_j^{\mathrm{T}} \lambda(t+jT), & -\infty < t < -NT \\ \gamma^2 u(t) - \sum_{j=i+1}^{N} D_j^{\mathrm{T}} \sum_{k=j-i}^{N} D_k u(t+(j-k)T) + \sum_{j=0}^{N} B_j^{\mathrm{T}} \lambda(t+jT) & \\ \quad - \sum_{j=i+1}^{N} D_j^{\mathrm{T}} C x(t+jT), & -(i+1)T < t < -iT \end{cases} \tag{5.4a}$$

$$\dot{x}(t) = \begin{cases} Ax(t) + \sum_{j=0}^{N} B_j u(t-jT), & -\infty < t < 0 \\ Ax(t) + \sum_{j=i+1}^{N} B_j u(t-jT), & iT < t < (i+1)T \\ Ax(t), & NT < t < \infty \end{cases} \tag{5.4b}$$

$$\dot{\lambda}(t) = \begin{cases} -A^{\mathrm{T}} \lambda(t), & -\infty < t < 0 \\ -A^{\mathrm{T}} \lambda(t) + C^{\mathrm{T}} C x(t) + \sum_{k=i+1}^{N} C^{\mathrm{T}} D_k u(t-kT), & iT < t < (i+1)T \\ -A^{\mathrm{T}} \lambda(t) + C^{\mathrm{T}} C x(t), & NT < t < \infty \end{cases} \tag{5.4c}$$

with the boundary condition $\lambda(\infty) = 0$ and free $\lambda(-\infty)$. Define a new variable $z(t) =$

$-\lambda(t)/\gamma$, then the equation for the variable $z$ is

$$0 = \begin{cases} \gamma^2 u(t) - \sum_{j=0}^{N} B_j^{\mathrm{T}} z(t+jT), & -\infty < t < -NT \\ \gamma^2 u(t) - \sum_{j=i+1}^{N} D_j^{\mathrm{T}} \sum_{k=j-i}^{N} D_k u(t+(j-k)T) - \sum_{j=0}^{N} B_j^{\mathrm{T}} z(t+jT) \\ \quad - \sum_{j=i+1}^{N} D_j^{\mathrm{T}} C x(t+jT), & -(i+1)T < t < -iT \end{cases} \tag{5.4d}$$

$$\dot{z}(t) = \begin{cases} -A^{\mathrm{T}} z(t), & -\infty < t < 0 \\ -A^{\mathrm{T}} z(t) - \frac{1}{\gamma} C^{\mathrm{T}} C x(t) - \frac{1}{\gamma} \sum_{k=i+1}^{N} C^{\mathrm{T}} D_k u(t-kT), & iT < t < (i+1)T \\ -A^{\mathrm{T}} z(t) - \frac{1}{\gamma} C^{\mathrm{T}} C x(t), & NT < t < \infty \end{cases} \tag{5.4e}$$

where $i = 0, 1, \ldots, N-1$ and the associated boundary condition $z(\infty) = 0$, and free $z(-\infty)$. In addition, we need to prove $\delta^2 J(x_*, u_*; \gamma) \geq 0$.

Let $x_*$, $u_*$ denote the optimal state and control which satisfy $\delta J(x_*, u_*; \gamma) = 0$. By direct computation, the second variation of $J_a$ is

$$\begin{aligned} \delta^2 J_a(u, x, \lambda; \gamma) &= \frac{\gamma^2}{2} \int_{-\infty}^{0} \delta u(t)^T \delta u(t) dt - \frac{1}{2} \int_{-\infty}^{0} \delta x(t)^T C^T C \delta x(t) dt \\ &- \sum_{i=0}^{N-1} \int_{iT}^{(i+1)T} \left( C\delta x(t) + \sum_{j=i+1}^{N} D_j \delta u(t-jT) \right)^T \left( C\delta x(t) + \sum_{k=i+1}^{N} D_k \delta u(t-kT) \right) dt \end{aligned}$$

where $\delta x$ and $\delta u$ must satisfy

$$\delta \dot{x}(t) = A\delta x(t) + \sum_{j=0}^{N} B_j \delta u(t-jT), \quad \delta x(0) = 0$$

Comparing with (5.3), the second variation $\delta^2 J_a$ can be expressed in terms of $J$, i.e.

$$\delta^2 J_a(u, x, \lambda; \gamma) = J(\delta u, \delta x; \gamma)$$

thus

$$\delta^2 J_a(u_*, x_*, \lambda; \gamma) = J(\delta u_*, \delta x_*; \gamma)$$

Since $J(u, x; \gamma) \geq 0$, for all $x$ and $u$, it follows that

$$\delta^2 J_a(u_*, x_*, \lambda; \gamma) = \delta^2 J(u_*, x_*; \gamma) \geq 0$$

### 5.1.1 Solution technique

Before computing the solutions of (5.4), we observe that

1. The time domain $(-\infty, \infty)$ can be partitioned into three subsets: $(-\infty, -NT]$, $[-NT, NT]$, and $[NT, \infty)$.

2. From (5.4e), once the value of $z(t)$ at $t = 0$ is known, say $z(0) = z_0$, the solution of $z(t)$ for $t < 0$ can be determined.

3. The variables $u$ and $x$ for $t < 0$ will be obtained once we know the value $x(0) = x_0$ and the restriction of the function $u(\cdot)$ to $[-NT, 0]$.

Thus, we only need to compute the functions $x(\cdot)$ and $z(\cdot)$ in $[0, NT]$ and $[NT, \infty)$, and the function $u(\cdot)$ in $[-NT, 0]$. We assume for the moment that $x_0$ and $z_0$ are free nonzero parameters.

First, consider the time interval $[NT, \infty)$. The functions $x(\cdot)$ and $z(\cdot)$ must satisfy

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -\frac{1}{\gamma}C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

or equivalently,

$$\begin{aligned} x(t) &= e^{A(t-NT)}x(NT) \\ z(t) &= e^{A^{\mathrm{T}}(NT-t)}\left(z(NT) - \frac{1}{\gamma}\int_{NT}^{t} e^{A^{\mathrm{T}}(\tau-NT)}C^{\mathrm{T}}Ce^{A(\tau-NT)}d\tau x(NT)\right) \end{aligned}$$

In order to satisfy the boundary condition $z(\infty) = 0$,

$$z(NT) = \frac{1}{\gamma}\int_{NT}^{\infty} e^{A^{\mathrm{T}}(\tau-NT)}C^{\mathrm{T}}Ce^{A(\tau-NT)}d\tau x(NT) = \frac{1}{\gamma}Qx(NT) \tag{5.5}$$

where $Q$ is the obervability gramian of the system (5.1) without input delays, i.e.,

$$Q = \int_{0}^{\infty} e^{A^{\mathrm{T}}\tau}C^{\mathrm{T}}Ce^{A\tau}d\tau$$

Next, consider the functions $x(\cdot)$ and $z(\cdot)$ and in the interval $[0, NT]$ and $u(\cdot)$ in $[-NT, 0]$. By using lifting operation, make the change of variables $\xi_j(t) = x(t + (j-1)T)$, $\eta_j(t) = z(t + (j-1)T)$, and $u_j(t) = u(t - jT)$ where $j = 1, 2, .., N$ and $\xi_1(0) = x_0$, $\eta_1(0) = z_0$, $\xi_N(T) = x(NT)$, and $\eta_N(T) = z(NT)$. This process is showed in Figure 5.1.

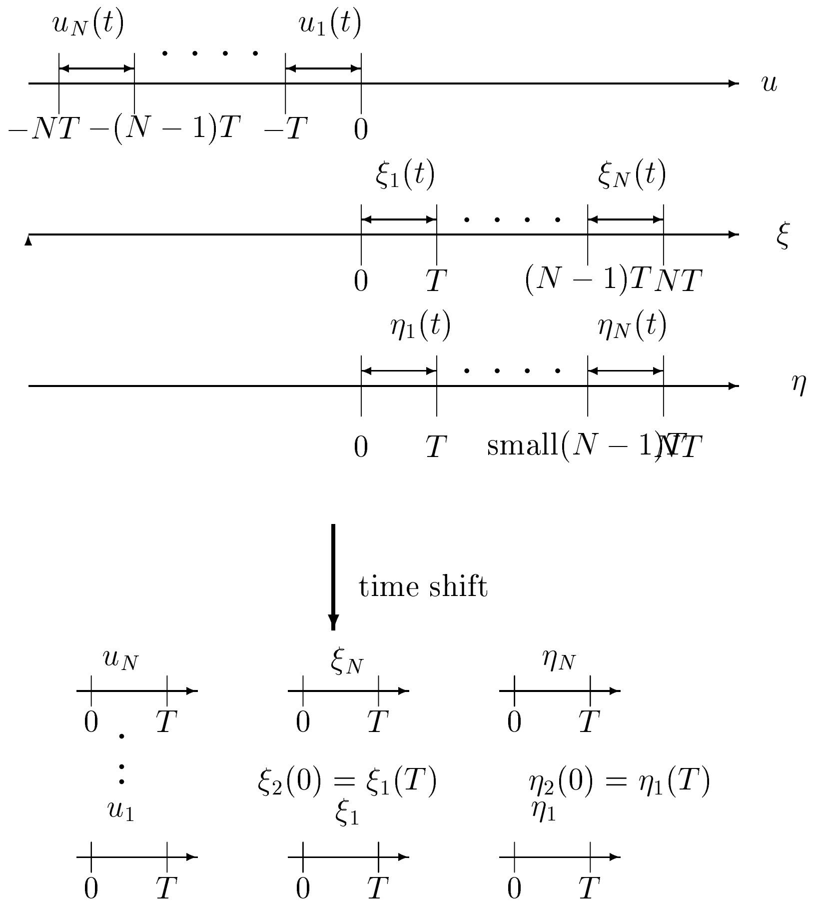



Figure 5.1: Lifting process on signals $u$, $\xi$, $\eta$

In this way, the equations in (5.4) are transformed to the following ones, defined on the same interval $[0, T]$:

$$\begin{cases} 0 = \gamma^2 u_1(t) - \gamma B_0^{\mathrm{T}} e^{-A^{\mathrm{T}}(t-T)}\eta_1(0) - \gamma \sum_{j=1}^{N} B_j^{\mathrm{T}}\eta_j(t) - \sum_{j=1}^{N} D_j^{\mathrm{T}} \sum_{k=j}^{N} D_k u_{k-j+1}(t) - \sum_{j=1}^{N} D_j^{\mathrm{T}} C\xi_j(t) \\ 0 = \gamma^2 u_2(t) - \gamma B_0^{\mathrm{T}} e^{-A^{\mathrm{T}}(t-2T)}\eta_1(0) - \gamma B_1^{\mathrm{T}} e^{-A^{\mathrm{T}}(t-T)}\eta_1(0) + \sum_{j=1}^{N-1} B_{j+1}^{\mathrm{T}}\eta_j(t) \\ \qquad - \sum_{j=2}^{N} D_j^{\mathrm{T}} \sum_{k=j-1}^{N} D_k u_{k-j+2}(t) - \sum_{j=2}^{N} D_j^{\mathrm{T}} C\xi_{j-1}(t) \\ \vdots \\ 0 = \gamma^2 u_N(t) - \gamma \sum_{j=0}^{N-1} B_j^{\mathrm{T}} e^{-A^{\mathrm{T}}(t-(N-j)T)}\eta_1(0) - \gamma B_N^{\mathrm{T}}\eta_1(t) - D_N^{\mathrm{T}} \sum_{k=1}^{N} D_k u_k(t) - D_N^{\mathrm{T}} C\xi_1(t) \end{cases} \tag{5.6a}$$

$$\begin{cases} \dot{\xi}_1(t) = A\xi_1(t) + B_1u_1(t) + B_2u_2(t) + \cdots + B_Nu_N(t) \\ \dot{\xi}_2(t) = A\xi_2(t) + B_2u_1(t) + B_3u_2(t) + \cdots + B_Nu_{N-1}(t) \\ \qquad \vdots \\ \dot{\xi}_N(t) = A\xi_N(t) + B_Nu_1(t) \end{cases} \tag{5.6b}$$

$$\begin{cases} \dot{\eta}_1(t) = -A^{\mathrm{T}}\eta_1(t) - \frac{1}{\gamma}C^{\mathrm{T}}C\xi_1(t) - \frac{1}{\gamma}C^{\mathrm{T}}D_1u_1(t) - \cdots - \frac{1}{\gamma}C^{\mathrm{T}}D_Nu_N(t) \\ \dot{\eta}_2(t) = -A^{\mathrm{T}}\eta_2(t) - \frac{1}{\gamma}C^{\mathrm{T}}C\xi_2(t) - \frac{1}{\gamma}C^{\mathrm{T}}D_2u_1(t) - \cdots - \frac{1}{\gamma}C^{\mathrm{T}}D_Nu_{N-1}(t) \\ \qquad \vdots \\ \dot{\eta}_N(t) = -A^{\mathrm{T}}\eta_N(t) - \frac{1}{\gamma}C^{\mathrm{T}}C\xi_N(t) - \frac{1}{\gamma}C^{\mathrm{T}}D_Nu_1(t) \end{cases} \tag{5.6c}$$

with the following initial conditions

$$\begin{cases} \xi_{10} \triangleq \xi_1(0) = x_0,\ \xi_{20} \triangleq \xi_2(0) = \xi_1(T), \cdots,\ \xi_{N0} \triangleq \xi_N(0) = \xi_{N-1}(T) \\ \eta_{10} \triangleq \eta_1(0) = z_0,\ \eta_{20} \triangleq \eta_2(0) = z_1(T), \cdots,\ \eta_{N0} \triangleq \eta_N(0) = \eta_{N-1}(T) \end{cases} \tag{5.6d}$$

Introducing vector notation for variables $u_i$, $\xi_i$, and $\eta_i$ as

$$\mathbf{u} = [u_1,\ u_2,\ \cdots,\ u_N]^{\mathrm{T}},\ \boldsymbol{\xi} = [\xi_1,\ \xi_2,\ \cdots,\ \xi_N]^{\mathrm{T}},\ \boldsymbol{\eta} = [\eta_1,\ \eta_2,\ \cdots,\ \eta_N]^{\mathrm{T}}$$

the system (5.6) is converted into differential-algebraic equations (DAE) of the form:

$$0 = \begin{bmatrix}\mathcal{D}^{\mathrm{T}} \otimes C & \gamma\mathcal{B}^{\mathrm{T}}\end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} - (\gamma^2 I - \mathcal{D}^{\mathrm{T}}\mathcal{D})\mathbf{u}(t) + \begin{bmatrix}0 & \gamma\mathcal{B}_0(t)^{\mathrm{T}}\end{bmatrix} \begin{bmatrix} \xi_{10} \\ \eta_{10} \end{bmatrix} \tag{5.7a}$$

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} = \begin{bmatrix} I \otimes A & 0 \\ -\frac{1}{\gamma}I \otimes C^{\mathrm{T}}C & I \otimes (-A)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ -\frac{1}{\gamma}C^{\mathrm{T}} \otimes \mathcal{D} \end{bmatrix} \mathbf{u}(t) \tag{5.7b}$$

where the matrices $\mathcal{B}$ and $\mathcal{B}_0(t)$ are defined as

$$\begin{aligned}
\mathcal{B} &= \begin{bmatrix} B_1 & \cdots & B_N \\ \vdots & & \vdots \\ B_N & \cdots & 0 \end{bmatrix}, \\
\mathcal{B}_0(t) &= \begin{bmatrix} e^{-A(t-T)}B_0 & e^{-A(t-2T)}B_0 + e^{-A(t-T)}B_1 & \cdots & \sum_{j=0}^{N-1} e^{-A[t-(N-j)T]}B_j \end{bmatrix}
\end{aligned}$$

and $\mathcal{D}$ is defined in (4.9). Let $\mathcal{D}_\gamma = \gamma^2 I - \mathcal{D}^{\mathrm{T}}\mathcal{D}$ and solving the function $\mathbf{u}(\cdot)$ from (5.7a) leads to

$$\mathbf{u}(t) = \mathcal{D}_\gamma^{-1}\begin{bmatrix}\mathcal{D}^{\mathrm{T}} \otimes C & \gamma\mathcal{B}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t)\end{bmatrix} + \mathcal{D}_\gamma^{-1}\begin{bmatrix}0 & \gamma\mathcal{B}_0(t)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\xi_{10} \\ \eta_{10}\end{bmatrix} \tag{5.8}$$

The substitution of $\mathbf{u}(\cdot)$ from (5.8) into (5.7b) gives us

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix}\boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t)\end{bmatrix} &= \left(\begin{bmatrix} I \otimes A & 0 \\ -\frac{1}{\gamma}I \otimes C^{\mathrm{T}}C & I \otimes (-A)^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ -\frac{1}{\gamma}C^{\mathrm{T}} \otimes \mathcal{D} \end{bmatrix}\mathcal{D}_\gamma^{-1}\begin{bmatrix}\mathcal{D}^{\mathrm{T}} \otimes C & \gamma\mathcal{B}^{\mathrm{T}}\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t)\end{bmatrix} \\
&\quad + \begin{bmatrix} \mathcal{B} \\ -\frac{1}{\gamma}C^{\mathrm{T}} \otimes \mathcal{D} \end{bmatrix}\mathcal{D}_\gamma^{-1}\begin{bmatrix}0 & \gamma\mathcal{B}_0(t)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\xi_{10} \\ \eta_{10}\end{bmatrix} \\
&\triangleq A_\gamma\begin{bmatrix}\boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t)\end{bmatrix} + B_\gamma(t)\begin{bmatrix}\xi_{10} \\ \eta_{10}\end{bmatrix}
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
A_\gamma &= \begin{bmatrix} I \otimes A + \mathcal{B}\mathcal{D}_\gamma^{-1}\mathcal{D}^{\mathrm{T}} \otimes C & \gamma\mathcal{B}\mathcal{D}_\gamma^{-1}\mathcal{B}^{\mathrm{T}} \\ -\frac{1}{\gamma}(I \otimes C^{\mathrm{T}}C + C^{\mathrm{T}} \otimes \mathcal{D}\mathcal{D}_\gamma^{-1}\mathcal{D}^{\mathrm{T}} \otimes C) & -I \otimes A^{\mathrm{T}} - C^{\mathrm{T}} \otimes \mathcal{D}\mathcal{D}_\gamma^{-1}\mathcal{B}^{\mathrm{T}} \end{bmatrix} \\
B_\gamma(t) &= \begin{bmatrix} 0 & \gamma\mathcal{B}\mathcal{D}_\gamma^{-1}\mathcal{B}_0(t)^{\mathrm{T}} \\ 0 & -C^{\mathrm{T}} \otimes \mathcal{D}\mathcal{D}_\gamma^{-1}\mathcal{B}_0(t)^{\mathrm{T}} \end{bmatrix}
\end{aligned}$$

We note that $A_\gamma$ is a Hamitonian matrix. The solution of the system(5.9) is then given by

$$\begin{bmatrix}\boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t)\end{bmatrix} = e^{A_\gamma t}\begin{bmatrix}\boldsymbol{\xi}(0) \\ \boldsymbol{\eta}(0)\end{bmatrix} + \int_0^t e^{A_\gamma(t-\tau)}B_\gamma(\tau)d\tau\begin{bmatrix}\xi_{10} \\ \eta_{10}\end{bmatrix} \tag{5.10}$$

From (5.6d) the relation between $[\xi_{10} \quad \eta_{10}]^{\mathrm{T}}$, $[\boldsymbol{\xi}(0) \quad \boldsymbol{\eta}(0)]^{\mathrm{T}}$, and $[\boldsymbol{\xi}(T) \quad \boldsymbol{\eta}(T)]^{\mathrm{T}}$ is described by

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\xi}(0)\\ \boldsymbol{\eta}(0)\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\\ 0 & 1\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi_{10}\\ \eta_{10}\end{bmatrix} + \begin{bmatrix}0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 1 & \cdots & 0 & \vdots & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix}\\ &\triangleq R_0\begin{bmatrix}\xi_{10}\\ \eta_{10}\end{bmatrix} + R_N\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix} \qquad (5.11)\end{aligned}$$

Thus, substituting (5.9) into (5.8) leads to

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} &= \left(e^{A_\gamma t}R_0 + \int_0^t e^{A_\gamma (t-\tau)}B_\gamma(\tau)d\tau\right)\begin{bmatrix}\xi_{10}\\ \eta_{10}\end{bmatrix} + e^{A_\gamma t}R_N\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix}\\ &\triangleq e^{A_\gamma t}\left(C_\gamma(t)\begin{bmatrix}\xi_{10}\\ \eta_{10}\end{bmatrix} + R_N\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix}\right) \qquad (5.12)\end{aligned}$$

When $t = T$ in (5.12), we obtain

$$\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix} = e^{A_\gamma T}\left(C_\gamma(T)\begin{bmatrix}\xi_{10}\\ \eta_{10}\end{bmatrix} + R_N\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix}\right)$$

or equivalently,

$$\begin{bmatrix}\boldsymbol{\xi}(T)\\ \boldsymbol{\eta}(T)\end{bmatrix} = \left(I - e^{A\gamma T}R_N\right)^{-1} e^{A\gamma T}C_\gamma(T)\begin{bmatrix}x_0\\ z_0\end{bmatrix}$$

Thus the functions $\boldsymbol{\xi}(\cdot)$ and $\boldsymbol{\eta}(\cdot)$ are given by

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} &= e^{A\gamma t}\left(C_\gamma(t) + R_N\left(I - e^{A\gamma T}R_N\right)^{-1} e^{A\gamma T}C_\gamma(T)\right)\begin{bmatrix}\xi_{10}\\ \eta_{10}\end{bmatrix}\\ &\triangleq \varphi(t)\begin{bmatrix}x_0\\ z_0\end{bmatrix} \qquad (5.13)\\ &= \begin{bmatrix}\varphi_{\xi x}(t) & \varphi_{\xi z}(t)\\ \varphi_{\eta x}(t) & \varphi_{\eta z}(t)\end{bmatrix}\begin{bmatrix}x_0\\ z_0\end{bmatrix}\end{aligned}$$

in which the matrices $\varphi(t)$ has the form

$$\varphi(t) = e^{A_\gamma t} R_0 + \int_0^t e^{A_\gamma (t-\tau)} B_\gamma(\tau) d\tau + e^{A_\gamma t} R_N \left(I - e^{A_\gamma T} R_N\right)^{-1} \left(e^{A_\gamma T} R_0 + \int_0^T e^{A_\gamma (T-\tau)} B_\gamma(\tau) d\tau\right) \quad (5.14a)$$

$$\varphi(T) = \left(I - e^{A_\gamma T} R_N\right)^{-1} \left(e^{A_\gamma T} R_0 + \int_0^T e^{A_\gamma (T-\tau)} B_\gamma(\tau) d\tau\right) \quad (5.14b)$$

Therefore, the relationship between $[x(NT),\ z(NT)]$ and $[x_0,\ z_0]$ is

$$\begin{bmatrix} x(NT) \\ z(NT) \end{bmatrix} = \begin{bmatrix} \xi_N(T) \\ \eta_N(T) \end{bmatrix} = \begin{bmatrix} (\varphi_{\xi x}(T))_N & (\varphi_{\xi z}(T))_N \\ (\varphi_{\eta x}(T))_N & (\varphi_{\eta z}(T))_N \end{bmatrix} \begin{bmatrix} \xi_{10} \\ \eta_{10} \end{bmatrix} \quad (5.15)$$

Finally, consider the time interval $[-NT, 0]$ to find the relation between $x_0$ and $z_0$. From (5.4e) we get

$$z(t) = e^{-A^{\mathrm{T}} t} z_0, \quad t < 0. \quad (5.16)$$

Since the function $\mathbf{u}(\cdot)$ can be obtained by substituting the functions $\boldsymbol{\xi}(\cdot)$ and $\boldsymbol{\eta}(\cdot)$ from (5.11) into (5.9) which leads to

$$\mathbf{u}(t) = \mathcal{D}_\gamma^{-1} \left( \left[\mathcal{D}^{\mathrm{T}} \otimes C \quad \gamma \mathcal{B}^{\mathrm{T}}\right] \varphi(t) + \left[0 \quad \gamma \mathcal{B}_0(t)^{\mathrm{T}}\right] \right) \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \triangleq \phi(t) \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \quad (5.17)$$

and then the restriction of the functions $u(\cdot)$ on $[-NT, 0]$ is given by the relation $u(t - jT) = u_j(t) = (\mathbf{u}(t))_j$. In this way, for $t < -NT$ the function $u(\cdot)$ is computed from (5.4a)

$$u(t) = -\frac{1}{\gamma^2} \sum_{j=0}^{N} B_j^{\mathrm{T}} \lambda(t + jT) = \frac{1}{\gamma} \sum_{j=0}^{N} B_j^{\mathrm{T}} z(t + jT) = \frac{1}{\gamma} \sum_{j=0}^{N} B_j^{\mathrm{T}} e^{-A^{\mathrm{T}}(t+jT)} z_0 \quad (5.18)$$

Since the solution of the state equation in the system (5.1) has the expression

$$x(t) = \int_{-\infty}^{t} e^{-A(\tau - t)} \sum_{j=0}^{N} B_j u(\tau - jT) d\tau$$

it follows that

$$\begin{aligned}
x_0 &= \int_{-\infty}^{0} e^{-A\tau} \sum_{j=0}^{N} B_j u(\tau - jT) d\tau = \sum_{j=0}^{N} \int_{-\infty}^{-jT} e^{-A(\tau+jT)} B_j u(\tau) d\tau \\
&= \int_{-\infty}^{-NT} \sum_{j=0}^{N} e^{-A(\tau+jT)} B_j u(\tau) d\tau + \sum_{k=1}^{N} \int_{-kT}^{-kT+T} e^{-A\tau} \sum_{j=0}^{k-1} B_j u(\tau) d\tau \\
&= e^{ANT} \int_{-\infty}^{0} \sum_{j=0}^{N} e^{-A(\tau+jT)} B_j u(\tau - NT) d\tau + \sum_{k=1}^{N} e^{AkT} \int_{0}^{T} e^{-A\tau} \sum_{j=0}^{k-1} B_j u(\tau - kT) d\tau \\
&= \frac{1}{\gamma} e^{ANT} \int_{-\infty}^{0} e^{-A\tau} \left( \sum_{j=0}^{N} e^{-AjT} B_j \right) \left( \sum_{i=0}^{N} B_i^{\mathrm{T}} e^{-A^{\mathrm{T}} iT} \right) e^{-A^{\mathrm{T}}\tau} d\tau \; e^{A^{\mathrm{T}} NT} z_0 \\
&\quad + \int_{0}^{T} e^{-A\tau} \sum_{k=1}^{N} e^{AkT} \sum_{j=0}^{k-1} B_j u_k(\tau) d\tau \\
&= \frac{1}{\gamma} P_N z_0 + \int_{0}^{T} e^{-A\tau} \sum_{k=1}^{N} e^{AkT} \left( \sum_{j=0}^{k-1} B_j \right) \left[ (\phi(\tau))_{k,1} x_0 + (\phi(\tau))_{k,2} z_0 \right] d\tau \\
&= \Psi_x x_0 + \left( \frac{1}{\gamma} P_N + \Psi_z \right) z_0 \qquad (5.19)
\end{aligned}$$

in which the matrices $P_N$, $\Psi_x$, and $\Psi_z$ are explicitly given by

$$\begin{aligned}
P_N &= \int_{-\infty}^{0} e^{-A\tau} \mathcal{B}(A) \mathcal{B}(A)^T e^{-A^T\tau} d\tau \\
\Psi_x &= \int_{0}^{T} e^{-A\tau} \sum_{k=1}^{N} e^{AkT} \sum_{j=0}^{k-1} B_j \; (\phi(\tau))_{k,1} d\tau \\
\Psi_z &= \int_{0}^{T} e^{-A\tau} \sum_{k=0}^{N-1} e^{AkT} \sum_{j=0}^{k-1} B_j (\phi(\tau))_{k,2} d\tau
\end{aligned}$$

Combining (5.15) and (5.19) through the relationship (5.15), we therefore obtain

$$\left( \begin{bmatrix} \frac{1}{\gamma} Q & -I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\varphi_{\xi x}(T))_N & (\varphi_{\xi z}(T))_N \\ (\varphi_{\eta x}(T))_N & (\varphi_{\eta z}(T))_N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -I + \Psi_x & \frac{1}{\gamma} P_N + \Psi_z \end{bmatrix} \right) \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = 0$$

In order to have nonzero solution for $[x_0, z_0]$, the following relationship must hold:

$$\det \left( \begin{bmatrix} \frac{1}{\gamma} Q & -I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\varphi_{\xi x}(T))_N & (\varphi_{\xi z}(T))_N \\ (\varphi_{\eta x}(T))_N & (\varphi_{\eta z}(T))_N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -I + \Psi_x & \frac{1}{\gamma} P_N + \Psi_z \end{bmatrix} \right) = 0 \qquad (5.20)$$

### 5.1.2 Main results

We can recapitulate

**Theorem 5.2** *Let $\gamma$ be the Hankel operator norm of the system (5.3) which is larger than $\rho_{ess}$. But then $\gamma$ is the largest nonzero solution of the equation (5.20).*

In the mean while, although the size of the matrix in equation (5.20) is $2n \times 2n$ which depends only on the size of the state space, the computation complexity of $P_N$, $\Psi_x$, and $\Psi_z$ increases as the number of delays increases. And when $N > 1$ it is very difficult to express the equation (5.20) into an algebraic equation of $\gamma$. Thus numerical method should be adopted to find the solution. So we develop a MATLAB program as given in Appendix to compute the Hankel norm of stable LTI systems using this Theorem 5.2.

## 5.2 Hankel singular vectors and values

Substituting the opoerators $\Gamma$ and $\Gamma^*$ from (4.3) and (4.4) into equation (5.2) leads to

$$(\Gamma^* v)(t) = (\sigma u)(t) = \int_0^\infty B_0^{\mathrm{T}} e^{A^{\mathrm{T}}(\tau - t)} C^{\mathrm{T}} v(\tau) d\tau +$$
$$\begin{cases} \sum_{j=1}^{N} \int_{\max\{0, t+jT\}}^{\infty} \tilde{B}_j^{\mathrm{T}} e^{A^{\mathrm{T}}(\tau - t)} C^{\mathrm{T}} v(\tau) d\tau + \sum_{j=i}^{N} D_j^{\mathrm{T}} v(t + jT), \\ \qquad -iT < t < -(i-1)T, \; i = 1, 2, .., N \\ \sum_{j=1}^{N} \int_0^\infty \tilde{B}_j^{\mathrm{T}} e^{A^{\mathrm{T}}(\tau - t)} C^{\mathrm{T}} v(\tau) d\tau, \quad t \le -NT \end{cases} \tag{5.21}$$

$$(\Gamma u)(t) = (\sigma v)(t) = \int_{-\infty}^{0} C e^{A(t-\tau)} B_0 u(\tau) d\tau +$$
$$\begin{cases} \sum_{j=1}^{N} \int_{-\infty}^{\min\{jT, t\}} C e^{A(t-\tau)} B_j u(\tau - jT) d\tau + \sum_{j=i}^{N} D_j u(t - jT), \\ \qquad (i-1)T < t < iT, \; i = 1, 2, ..., N \\ \sum_{j=1}^{N} \int_{-\infty}^{jT} C e^{A(t-\tau)} B_j u(\tau - jT) d\tau, \quad t \ge NT \end{cases} \tag{5.22}$$

### 5.2.1 Solution technique

In order to solve the variable $\sigma$ and corresponding vectors $(u, v)$ from equations (5.21) and (5.22), we need to consider the functions $\xi_i(t)$, $\eta_i(t)$ defined on the interval $[0, T]$ first.

$$
\begin{aligned}
\xi_1(t) &= \int_{-\infty}^{0} e^{A(t-\tau)} B_0 u(\tau) d\tau + \int_{-\infty}^{t} e^{A(t-\tau)} \sum_{j=1}^{N} B_j u(\tau - jT) d\tau \\
\xi_i(t) &= \int_{-\infty}^{0} e^{A(t+(i-1)T-\tau)} B_0 u(\tau) d\tau + \sum_{j=1}^{i-1} \int_{-\infty}^{jT} e^{A(t+jT)} B_j u(\tau - jT) d\tau \\
&\quad + \int_{-\infty}^{t} e^{A(t-\tau)} \sum_{j=i}^{N} B_j u(\tau - (j-i)T) d\tau, \quad i = 2, 3, .., N \qquad (5.23) \\
\eta_i(t) &= \int_{t+(i-1)T}^{\infty} e^{A^T(\tau - t - (i-1)T)} C^T v(\tau) d\tau \\
&= \int_{t}^{\infty} e^{A^T(\tau - t)} C^T v(\tau + (i-1)T) d\tau, \quad i = 1, 2, .., N
\end{aligned}
$$

In terms of $\xi_i(t)$ and $\eta_i(t)$, $i = 1, 2, \ldots, N$, equations (5.21) and (5.22) can be rewritten as

$$
\sigma v(t) = \begin{cases} C\xi_i(t - (i-1)T) + \sum_{j=i}^{N} D_j u(t - jT), & (i-1)T < t < iT, \ i = 1, 2, ..., N \\ Ce^{A(t-NT)} \xi_N(T), & t \geq NT \end{cases} \qquad (5.24)
$$

$$
\sigma u(t) = B_0^{\mathrm{T}} e^{-A^{\mathrm{T}} t} \eta_1(0) + \begin{cases} \sum_{j=1}^{N} \left[ B_j^{\mathrm{T}} \eta_j(t+T) + D_j^{\mathrm{T}} v(t+jT) \right], & -T < t < 0 \\ \sum_{j=1}^{i-1} \tilde{B}_j^{\mathrm{T}} e^{-A^{\mathrm{T}} t} \eta_1(0) + \sum_{j=i}^{N} \left[ B_j^{\mathrm{T}} \eta_j(t+T) + D_j^{\mathrm{T}} v(t+jT) \right], & \\ \qquad -iT < t < -(i-1)T, \ i = 2, 3, ..., N & \\ \sum_{j=1}^{N} \tilde{B}_j^{\mathrm{T}} e^{-A^{\mathrm{T}} t} \eta_1(0), \ t \leq -NT & \end{cases} \qquad (5.25)
$$

It is easily seen that any solutions $\xi_N(T)$ and $\eta_1(0)$ in equations (5.24) and (5.25) can be obtained if the restrictions of the functions $\xi_i(\cdot)$ and $\eta_i(\cdot)$, $i = 1, 2, \ldots, N$, in the interval $[0, NT]$ are identified. It is convenient to make the change of variables by letting

$$
\begin{cases} u_j(t) = u(t - jT) \\ v_j(t) = v(t + (j-1)T) \end{cases}
$$

defined on the interval $[0, T]$ where $j = 1, 2, .., N$. The process is showed in Figure 5.2.

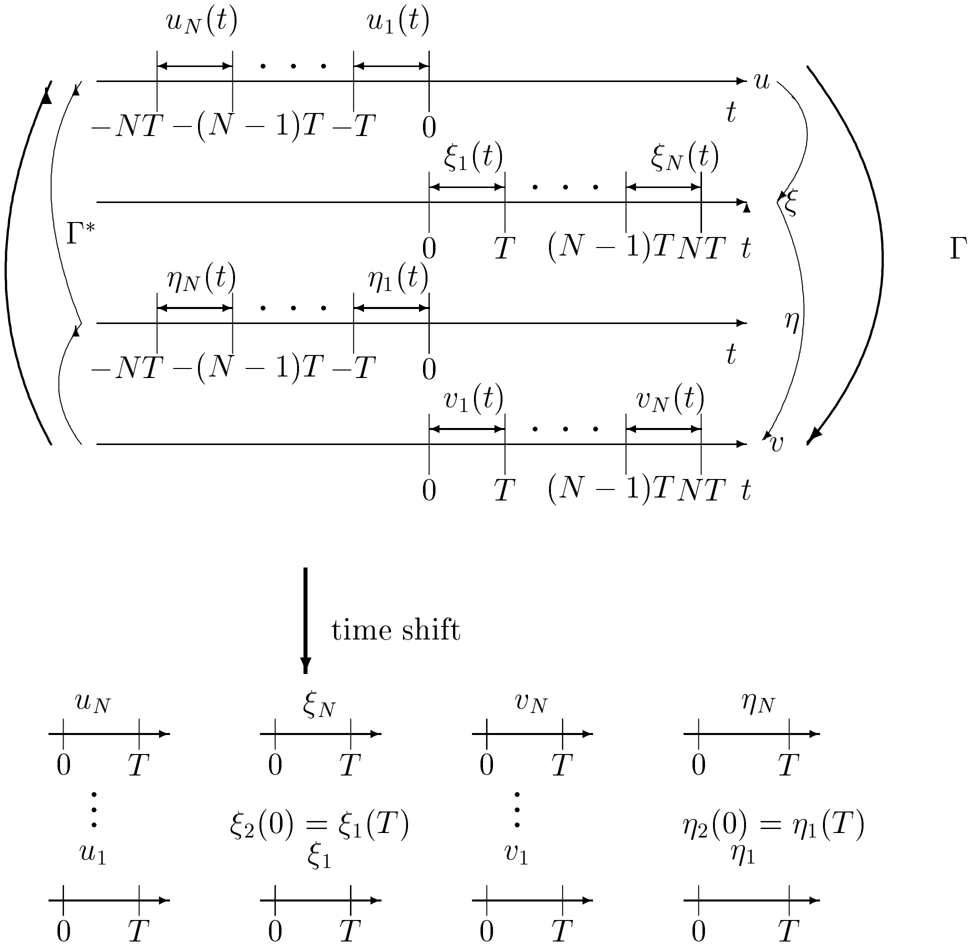



Figure 5.2: Time interval's partition and shifting for signal $u$, $v$, $\xi$, and $\eta$

Now the system (5.24) and (5.25) to be studied in the interval $[0, NT]$ is changed to a new system defined only in the interval $[0, T]$ given by

$$\begin{aligned} \sigma v_i(t) &= C\xi_i(t) + \sum_{j=i}^{N} D_j u_{i-j+1}(t) \\ \sigma u_i(t) &= \sum_{j=0}^{i-1} B_j^T e^{-A^T(t-(i-j)T)}\eta_{10} + \sum_{j=i}^{N} \left[B_j^T \eta_{j-i+1}(t) + D_j^T v_{j-i+1}(t)\right] \end{aligned} \tag{5.26}$$

with $i = 1, 2, .., N$. And the variables $(\xi_i(t), \eta_i(t))$ in equation (5.23) can be expressed in terms of the variables $u_i(t)$ and $v_i(t)$ to the form of differential equations:

$$\left\{\begin{array}{lcl} \dot{\xi}_1(t) &=& A\xi_1(t) + B_1u_1(t) + B_2u_2(t) + \cdots + B_Nu_N(t) \\ \dot{\xi}_2(t) &=& A\xi_2(t) + B_2u_1(t) + B_3u_2(t) + \cdots + B_Nu_{N-1}(t) \\ &\vdots& \\ \dot{\xi}_N(t) &=& A\xi_N(t) + B_Nu_1(t) \\ \dot{\eta}_1(t) &=& -A^T\eta_1(t) - C^Tv_1(t) \\ \dot{\eta}_2(t) &=& -A^T\eta_2(t) - C^Tv_2(t) \\ &\vdots& \\ \dot{\eta}_N(t) &=& -A^T\eta_N(t) - C^Tv_N(t) \end{array}\right. \tag{5.27a}$$

with the following initial conditions

$$\left\{\begin{array}{l} \xi_{10} \triangleq \xi_1(0) = \sum_{j=0}^{N} \int_{-\infty}^{0} e^{-A\tau} B_j u(\tau - jT)d\tau,\ \xi_{20} \triangleq \xi_2(0) = \xi_1(T), \cdots,\ \xi_{N0} \triangleq \xi_N(0) = \xi_{N-1}(T) \\ \eta_{10} \triangleq \eta_1(0) = \int_{0}^{\infty} e^{A^T\tau} C^T C v(\tau)d\tau,\ \eta_{20} \triangleq \eta_2(0) = \eta_1(T), \cdots,\ \eta_{N0} \triangleq \eta_N(0) = \eta_{N-1}(T) \end{array}\right. \tag{5.27b}$$

For simplicity, we introduce vector notation for variables $u_i$, $v_i$, $\xi_i$, and $\eta_i$ as following

$$\mathbf{u} = \begin{bmatrix} u_1, u_2, \cdots, u_N \end{bmatrix}^T, \mathbf{v} = \begin{bmatrix} v_1, v_2, \cdots, v_N \end{bmatrix}^T, \boldsymbol{\xi} = \begin{bmatrix} \xi_1, \xi_2, \cdots, \xi_N \end{bmatrix}^T, \boldsymbol{\eta} = \begin{bmatrix} \eta_1, \eta_2, \cdots, \eta_N \end{bmatrix}^T$$

and use the vectors $\boldsymbol{\xi}_0$ and $\boldsymbol{\eta}_0$ denote the corresponding initial values of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ at $t = 0$, then the systems (5.26) and (5.27) form a system of differential-algebraic equations (DAE) with the following vector representation :

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} = \begin{bmatrix}I\otimes A & 0\\ 0 & I\otimes(-A)^T\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} + \begin{bmatrix}0 & \mathcal{B}\\ I\otimes(-C)^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{u}(t)\end{bmatrix} \tag{5.28a}$$

$$0 = \begin{bmatrix}I\otimes C & 0\\ 0 & \mathcal{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} + \begin{bmatrix}-\sigma I & \mathcal{D}\\ \mathcal{D}^T & -\sigma I\end{bmatrix}\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{u}(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \widehat{\mathcal{B}}_0(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_0\\ \boldsymbol{\eta}_0\end{bmatrix} \tag{5.28b}$$

where the matrix $\mathcal{B}$ and $\widehat{\mathcal{B}}_0(t)$ are defined as

$$\mathcal{B} = \begin{bmatrix}B_1 & \cdots & B_N\\ \vdots & & \vdots\\ B_N & \cdots & 0\end{bmatrix}, \quad \widehat{\mathcal{B}}_0(t) = \begin{bmatrix}\mathcal{B}_0^T(t) & 0\end{bmatrix}$$

Firstly, we solve $\mathbf{u}$ and $\mathbf{v}$ from (5.28b)

$$\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{u}(t)\end{bmatrix} = \mathcal{D}_\sigma^{-1}\begin{bmatrix}I\otimes C & 0\\ 0 & \mathcal{B}^T\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} + \mathcal{D}_\sigma^{-1}\begin{bmatrix}0 & 0\\ 0 & \widehat{\mathcal{B}}_0(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_0\\ \boldsymbol{\eta}_0\end{bmatrix} \tag{5.29}$$

where the matrix $\mathcal{D}_\sigma$ are given by

$$\mathcal{D}_\sigma = \sigma I - \begin{bmatrix}0 & \mathcal{D}\\ \mathcal{D}^T & 0\end{bmatrix}$$

here $I$ denotes an $2N \times 2N$ identity matrix. The substitution of (5.29) into (5.28a) leads to

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} &= \left(\begin{bmatrix}I\otimes A & 0\\ 0 & I\otimes(-A)^T\end{bmatrix} + \begin{bmatrix}0 & \mathcal{B}\\ I\otimes(-C)^T & 0\end{bmatrix}\mathcal{D}_\sigma^{-1}\begin{bmatrix}I\otimes C & 0\\ 0 & \mathcal{B}^T\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}0 & \mathcal{B}\\ I\otimes(-C)^T & 0\end{bmatrix}\mathcal{D}_\sigma^{-1}\begin{bmatrix}0 & 0\\ 0 & \widehat{\mathcal{B}}_0(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_0\\ \boldsymbol{\eta}_0\end{bmatrix}\\ &\triangleq A_\sigma\begin{bmatrix}\boldsymbol{\xi}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} + B_\sigma(t)\begin{bmatrix}\boldsymbol{\xi}_0\\ \boldsymbol{\eta}_0\end{bmatrix}\end{aligned} \tag{5.30}$$

in which the matrices $A_\sigma$ and $B_\sigma$ have explicit forms as

$$\begin{aligned} A_\sigma &= \begin{bmatrix} I\otimes A & 0 \\ 0 & I\otimes(-A)^T \end{bmatrix} + \begin{bmatrix} 0 & \mathcal{B} \\ I\otimes(-C)^T & 0 \end{bmatrix} \mathcal{D}_\sigma^{-1} \begin{bmatrix} I\otimes C & 0 \\ 0 & \mathcal{B}^T \end{bmatrix} \\ B_\sigma &= \begin{bmatrix} 0 & \mathcal{B} \\ I\otimes(-C)^T & 0 \end{bmatrix} \mathcal{D}_\sigma^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \widehat{\mathcal{B}}_0(t) \end{bmatrix} \end{aligned}$$

The solution of the system(5.30) is then given by

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} &= \left( e^{A_\sigma t} + \int_0^t e^{A_\sigma (t-\tau)} B_\sigma(\tau) d\tau \right) \begin{bmatrix} \boldsymbol{\xi}_0 \\ \boldsymbol{\eta}_0 \end{bmatrix} \\ &\triangleq \varphi(\sigma, t) \begin{bmatrix} \boldsymbol{\xi}_0 \\ \boldsymbol{\eta}_0 \end{bmatrix} \end{aligned} \tag{5.31}$$

The restrictions of the functions $u_j(\cdot)$ and $v_j(\cdot)$ in $[0, T]$ can be obtained by substituting $\xi_j(t)$ and $\eta_j(t)$ from (5.31) into (5.29) which leads to

$$\begin{aligned} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{u}(t) \end{bmatrix} &= \mathcal{D}_\sigma^{-1} \left( \begin{bmatrix} I\otimes C & 0 \\ 0 & \mathcal{B}^T \end{bmatrix} \varphi(\sigma, t) + \begin{bmatrix} 0 & \mathbf{0} \\ 0 & \widehat{\mathcal{B}}_0(t) \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\xi}_0 \\ \boldsymbol{\eta}_0 \end{bmatrix} \\ &\triangleq \begin{bmatrix} \phi_v(t) \\ \phi_u(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_0 \\ \boldsymbol{\eta}_0 \end{bmatrix} \end{aligned} \tag{5.32}$$

Since by equation (5.27b) we have

$$\xi_i(T) = \xi_{i+1}(0), \quad \eta_i(T) = \eta_{i+1}(0) \tag{5.33}$$

for $i = 1, 2, \ldots, N-1$, and from equations (5.32) and (5.33) we can obtain

$$\begin{aligned} \eta_N(T) &= \eta(NT) \\ &= \int_{NT}^{\infty} e^{A^T(\tau - NT)} C^T v(\tau) d\tau \\ &= \frac{1}{\sigma} \int_{NT}^{\infty} e^{A^T(\tau - NT)} C^T C e^{A(\tau - NT)} \xi_N(T) d\tau \\ &= \frac{1}{\sigma} \int_0^{\infty} e^{A^T\tau} C^T C e^{A\tau} \xi_N(T) d\tau \\ &= \frac{1}{\sigma} Q \xi_N(T) \end{aligned} \tag{5.34}$$

where

$$Q = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau$$

Similarly, we also have

$$\begin{aligned}
\sigma\xi_{10} &= \sigma\xi_1(0) \\
&= \sigma \sum_{j=0}^{N} \int_{-\infty}^{0} e^{-A\tau} B_j u(\tau - jT) d\tau \\
&= \sigma \sum_{j=0}^{N} \int_{-\infty}^{-jT} e^{-A(\tau+jT)} B_j u(\tau) d\tau \\
&= \int_{-\infty}^{-NT} \sum_{j=0}^{N} e^{-A(\tau+jT)} B_j \sigma u(\tau) d\tau + \sigma \sum_{j=0}^{N} \int_{-(j+1)T}^{-jT} \sum_{i=0}^{j} e^{-A(\tau+iT)} B_i u(\tau) d\tau \\
&= \int_{-\infty}^{0} e^{-A\tau} \sum_{j=0}^{N} e^{A(N-j)T} B_j \sigma u(\tau - NT) d\tau \\
&\quad + \sigma \sum_{j=0}^{N-1} \int_0^T \sum_{i=0}^{j} e^{-A(\tau-(j+1-i)T)} B_i u_{j+1}(\tau) d\tau
\end{aligned} \tag{5.35}$$

and after substituting the $u_{j+1}(\cdot)$ in $[0, T]$ from (5.32) into equation (5.35) it follows that

$$\xi_{10} = \frac{1}{\sigma} P_N \eta_{10} + \Phi \begin{bmatrix} \boldsymbol{\xi_0} \\ \boldsymbol{\eta_0} \end{bmatrix} \tag{5.36}$$

where the matrices $P_N$ and $\Phi$ are given by

$$\begin{aligned}
P_N &= \int_{-\infty}^{0} e^{-A\tau} \mathcal{B}(A) \mathcal{B}(A)^T e^{-A^T\tau} d\tau \\
\Phi &= \int_0^T \begin{bmatrix} 0 & \mathcal{B}_0(t) \end{bmatrix} \phi(\tau) d\tau
\end{aligned}$$

Finally, from (5.33), (5.34), and (5.36), we obtain

$$\begin{bmatrix} 0 & 0 & -I & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & -I & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ (I-\Phi_1) & -(\frac{1}{\sigma}P_N + \Phi_{N+1}) & \cdots & \cdots & \cdots & -\Phi_N & -\Phi_{2N} \end{bmatrix} \begin{bmatrix} \xi_{10} \\ \eta_{10} \\ \xi_{20} \\ \eta_{20} \\ \vdots \\ \xi_{N0} \\ \eta_{N0} \end{bmatrix}$$
$$+ \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{\sigma}Q & I \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1(T) \\ \eta_1(T) \\ \vdots \\ \xi_N(T) \\ \eta_N(T) \end{bmatrix} = 0 \tag{5.37}$$

Define a permutation matrix $R$ such that

$$\begin{bmatrix} \xi_1(t) \\ \eta_1(t) \\ \xi_2(t) \\ \vdots \\ \eta_{N-1}(t) \\ \xi_N(t) \\ \eta_N(t) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \vdots \\ \eta_{N-2}(t) \\ \eta_{N-1}(t) \\ \eta_N(t) \end{bmatrix} = R \begin{bmatrix} \boldsymbol{\xi}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix}$$

for all $t$.

### 5.2.2 Main results

We can recapture our results as

**Theorem 5.3** *The singular values, $\sigma$, of $\Gamma$ are the nonzero solutions of the following equa-*

*tion*

$$\det\left(\begin{bmatrix} 0 & 0 & -I & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & -I & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ (I-\Phi_1) & -(\frac{1}{\sigma}P_N+\Phi_{N+1}) & \cdots & \cdots & \cdots & -\Phi_N & -\Phi_{2N} \end{bmatrix} R \right.$$

$$\left. + \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{\sigma}Q & I \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} R\varphi(\sigma,T) \right) = 0 \qquad (5.38)$$

Although the equation (5.20) and (5.38) have the different form, but by the corollary 5.1, we know that they have the same solutions.

## 5.3 Numerical examples

Before we compute the Hankel norm for delay systems, we need to identify the upper bound of Hankel norm. Once the upper bound is obtain, a bisection method can be applied to compute the value of Hankel norm. For a general Hankel operator, the upper bound of its norm is given by :

**Lemma 5.1** [5] *Suppose that* $h \in \mathcal{P}_1([0,\infty);\mathcal{L}(U,Y))$ *where* $U$ *and* $Y$ *are separable Hilbert spaces. The Hankel operator* $\Gamma_h$ *associated with* $h$ *is defined by*

$$\Gamma_h u(t) = \int_0^\infty h(t+\tau)u(\tau)d\tau$$

*for* $u \in \mathcal{L}([0,\infty);U)$*. Then* $\Gamma_h \in \mathcal{L}(\mathcal{L}_2([0,\infty);U),\mathcal{L}_2([0,\infty);Y)$ *with*

$$\|\Gamma_h\| \le \int_0^\infty \|h(t)\|_{\mathcal{L}(U,Y)}dt$$

In our study, the Hankel operator has the form as specified by (4.4) therefore its upper bound can be computed follows.

**Lemma 5.2** *The Hankel operator norm* $\Gamma$ *of the system (2.10) is bounded by*

$$\|\Gamma\| \le \frac{M}{\alpha}\|C\|\sum_{j=0}^{N}\|B_j\| + \sum_{j=1}^{N}\|D_j\|$$

*where* $M > 0, \alpha > 0$.

**Proof.**

$$\begin{aligned}
\|\Gamma\| &\leq \int_0^\infty \|h(t)\|_{\mathcal{L}(\mathcal{L}_-^2,\mathcal{L}_+^2)}dt \\
&\leq \int_0^\infty \left( \|\sum_{j=0}^N Ce^{A(t-jT)}B_jH(t-jT)\| + \|\sum_{j=1}^N D_j\delta(t-jT)\| \right) dt \\
&\leq \|C\| \sum_{j=0}^N \int_0^\infty \|e^{A(t-jT)}\|\|B_j\|H(t-jT)dt + \sum_{j=1}^N \int_0^\infty \|D_j\|\delta(t-jT)dt \\
&\leq \|C\| \sum_{j=0}^N \|B_j\| \int_{-jT}^\infty \|e^{A(t-jT)}\|dt + \sum_{j=1}^N \|D_j\|
\end{aligned}$$

Since $A$ is stable, by using Definition 2.1, there exist $M$, $\alpha > 0$ such that

$$\|e^{At}\| \leq Me^{-\alpha t}$$

Therefore

$$\begin{aligned}
\|\Gamma\| &\leq \|C\| \sum_{j=0}^N \|B_j\|M \int_{jT}^\infty e^{-\alpha(t-jT)}dt + \sum_{j=1}^N \|D_j\| \\
&= \frac{M}{\alpha}\|C\| \sum_{j=0}^N \|B_j\| + \sum_{j=1}^N \|D_j\|
\end{aligned}$$

■

The following examples are uses to illustrate the computation results of Hankel norm based on Theorem 5.2 and 5.3.

**Example 5.1** Let $G(s) = \dfrac{b_0 + b_1e^{-sT}}{s+1} + d_1e^{-sT}$ where $b_0, b_1$, and $d_1$ are constants. The state-space realization of $G(s)$ is given by

$$\begin{aligned}
\dot{x}(t) &= -x(t) + b_0u(t) + b_1u(t-T) \\
y(t) &= x(t) + d_1u(t-T)
\end{aligned}$$

**Case 1**: $b_0 = 1, b_1 = 0, d_1 = 1$. Theorem 5.2 will be applied to compute the Hankel norm. Comparing with the standard representation (5.1) leads to

$$A = -1,\ B_0 = 1,\ B_1 = 0,\ C = 1,\ D_1 = 1,\ T = 1$$

Since $A = -1$, we choose $M = 1$, $\alpha = 1$, and from Lemma 5.2, the upper bound for the norm of $\Gamma$ is

$$\|\Gamma\| \leq 2$$

By

$$A_\sigma = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -\dfrac{\sigma}{\sigma^2-1} & 0 & 1 & 0 \\ 0 & -\dfrac{1}{\sigma} & 0 & 1 \end{bmatrix}, \qquad B_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{e^{t-1}}{\sigma^2-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\varphi(\sigma,t) = \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ \dfrac{\sigma(e^{-t}-e^{t})}{2(\sigma^2-1)} & 0 & e^{t}-\dfrac{te^{t-1}}{\sigma^2-1} & 0 \\ 0 & \dfrac{(e^{-t}-e^{t})}{2(\sigma^2-1)} & 0 & e^{t} \end{bmatrix}, \quad \phi_u(t) = \begin{bmatrix} \dfrac{e^{-t}}{\sigma^2-1} & 0 & \dfrac{\sigma e^{t-1}}{\sigma^2-1} & 0 \\ 0 & 0 & \dfrac{1}{\sigma}e^{t-2} & 0 \end{bmatrix},$$

$$\Phi = \begin{bmatrix} \dfrac{e^{-1}}{\sigma^2-1} & 0 & -\dfrac{1}{2\sigma}\dfrac{-\sigma^2+e^{-2}+e^{-4}\sigma^2-e^{-4}}{\sigma^2-1} & 0 \end{bmatrix},$$

then the Hankel singular value $\sigma$ is determined by

$$\det\left(\begin{bmatrix} e^{-1} & -1 & 0 & 0 \\ -\dfrac{1}{2}\sigma\dfrac{-e^{-1}+e}{\sigma^2-1} & 0 & e-\dfrac{1}{\sigma^2-1} & -1 \\ 0 & -\dfrac{1}{2}\dfrac{e}{\sigma} & 0 & e \\ 1-\dfrac{1}{\sigma^2-1}e^{-1} & 0 & -\dfrac{1}{2}\dfrac{\sigma^2-e^{-2}}{(\sigma^2-1)\,\sigma} & 0 \end{bmatrix}\right) = 0$$

or equivalently,

$$\frac{1}{4}\left(6e^{-2}\sigma^2 - e^{-4} - 9\sigma^4 + 4\sigma^6 - 8\sigma^4e^{-1} + 4\sigma^2 + 8e^{-1}\sigma^2\right)\frac{e^2}{\sigma^2\left(\sigma^2-1\right)^2} = 0$$

The possible solutions are 0.0486869, 0.974666, and 1.42598. Since the essential spectrum is 1. Thus the Hankel norm of this system is 1.42598. Similiar to our study presented in case 1, the effect of delay times and $D_1$ on Hankel norm are considered in the following.

1. $D_1$ is fixed, but $T$ variates. The result as shown in Figure 5.3 states that the value of the Hankel norm $\|G\|_H$ approaches 1.29 for large delay time.

2. $T$ is fixed, but change the magnitude of $D_1$. The computational results, as depicted in Figure 5.4, is the value of $\|G\|_H$ is proportional to the magnitude of $D_1$ terms.

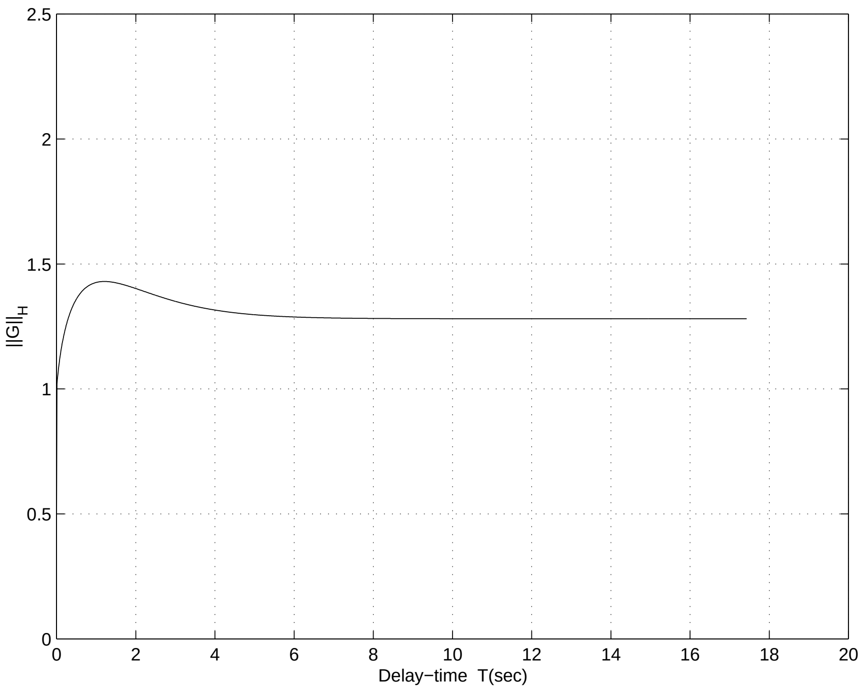



Figure 5.3: Variation of $\|G\|_H$ with different size of delay time $T$

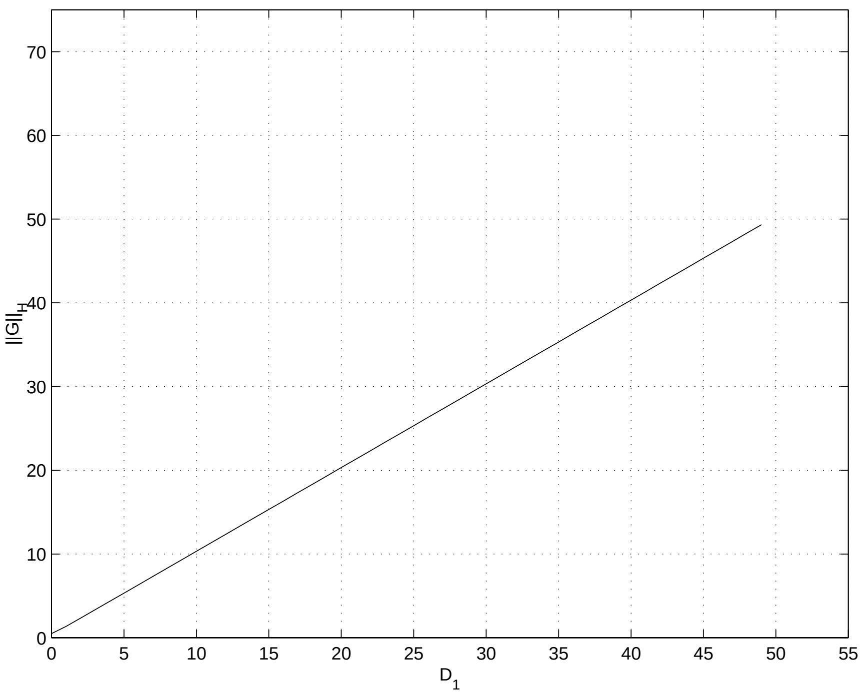



Figure 5.4: Effect of the magnitude $D_1$ on $\|G\|_H$

**Case 2** : $b_0 = 0$, $b_1 = 1$, $d_1 = 0$. Theorem 5.3 will be adopted. Comparing with (5.1) given as

$$A = -1,\ B_0 = 0,\ B_1 = C = 1,\ D_1 = 0,\ T = 1,\ N = 1$$

Similarly, the upper bound for the norm of $\Gamma$ is equal to 1. Let $N = 2$, adding two more coefficients $B_2 = 0$ and $D_2 = 0$. By (5.20), $D_\gamma = \gamma^2$, $R_0 = I$, $R_N = 0$, $B = B_1$, $\varphi(T) = e^{A_\gamma T}$, $P_1 = \frac{1}{2}e^2$, $Q = \frac{1}{2}$, and $\Psi_x = \Psi_z = 0$, then

$$\det\left(\begin{bmatrix} \frac{1}{2\gamma} & -1 \\ 0 & 0 \end{bmatrix} e^{A_\gamma T} + \begin{bmatrix} 0 & 0 \\ -1 & \frac{1}{2\gamma} \end{bmatrix}\right) = 0$$

which can be reduced to give $\tan(\lambda T) = \lambda(\lambda^2 - 3)/(1 - 3\lambda^2)$ where $\lambda^2 = \gamma^{-2} - 1$. With the aid of MATLAB code, the Hankel norms are listed in table:

| $T$ | 0 | 0.1 | 0.5 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $\|\|G\|\|_H$ | 0.5 | 0.542 | 0.656 | 0.737 | 0.826 | 0.876 |

Furthermore, asymptotical behavior of the Hankel norm for different size of $T$ is studied. As shown in Figure 5.5, there exists a limiting value (less than 1) of $||G||_H$ for large $T$.

In order to understand the effect of $D_1$ terms on the limiting value of $||G||_H$, we repeat previous numerical expeniment with different value of $d_1$, i.e. $D_1 = d_1 = 1$.

**Case 3** : $b_0 = 0$, $b_1 = 1$, $d_1 = 1$. Theorem 5.3 will be adopted. Comparing with (5.1) given as

$$A = -1,\ B_0 = 0,\ B_1 = C = 1,\ D_1 = 1,\ T = 1,\ N = 1$$

Similarly, the upper bound for the norm of $\Gamma$ is equal to 1. Whose results is given in Figure 5.6. And it is obviously that there still exists an limiting value of $||G||_H$ but with larger magnitude.

If $B_1 = 1,\ D_1 = 1$, $T \to \infty$, we can depict the change of determinate and Hankel norm in Figure 5.6.

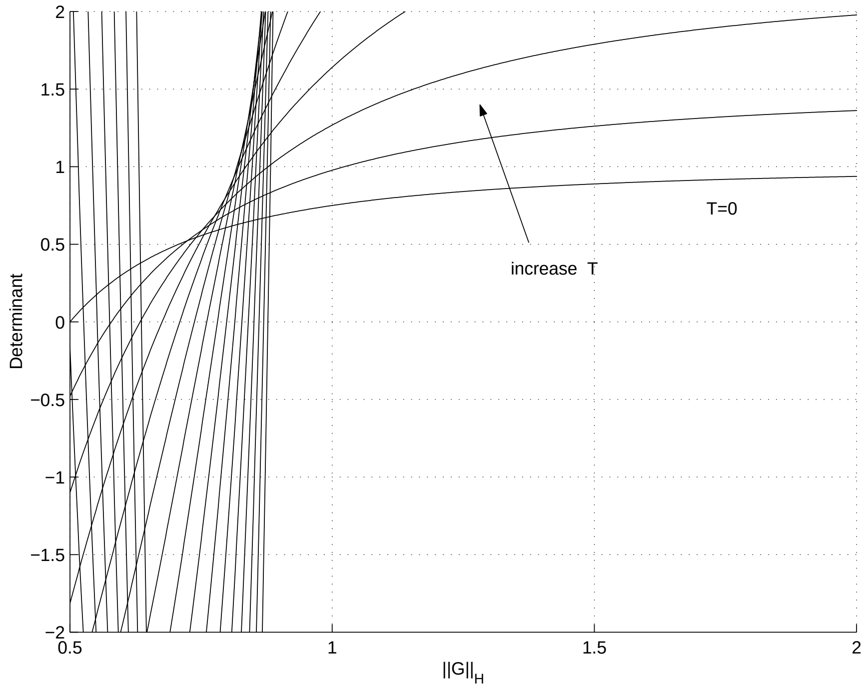



Figure 5.5: The asymptotic behavior of $\|G\|_H$ for large $T$

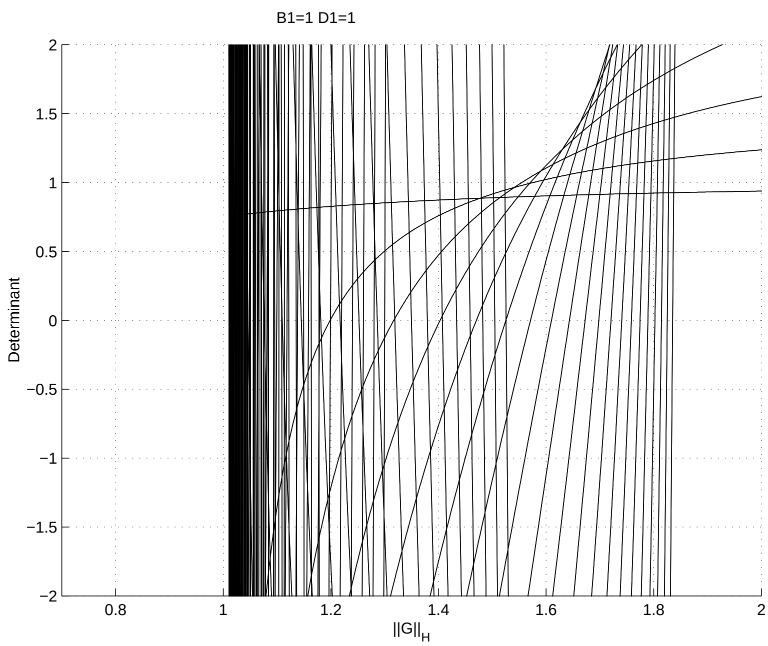



Figure 5.6: The asymptotic behavior of $\|G\|_H$ for large $T$

**Example 5.2** Given the system with two input delays :

$$\begin{aligned}\dot{x} &= -x(t) + u(t) + b_1\ u(t-1) + b_2\ u(t-2) \\ y(t) &= x(t) + u(t-1) + u(t-2)\end{aligned}$$

**Case 1**: $b_1 = b_2 = 0$. Only the effect of $D_j$ terms is studied.

Since $A = -1$, $B_0 = 1$, $B_1 = 0$, $B_2 = 0$, $C = 1$, $D_1 = 1$, $D_2 = 1$. By Lemma 5.2, the upper bound of Hankel norm is 3. Since

$$A_\sigma = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\sigma(\sigma^2-1)}{\hat{\sigma}} & -\frac{\sigma}{\hat{\sigma}} & 1 & 0 \\ -\frac{\sigma}{\hat{\sigma}} & -\frac{\sigma(\sigma^2-2)}{\hat{\sigma}} & 0 & 1 \end{bmatrix}, B_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sigma^2 e^{t-1}}{\hat{\sigma}} - \frac{(\sigma^2-1)e^{t-2}}{\hat{\sigma}} & 0 \\ 0 & 0 & -\frac{(\sigma^2-1)e^{t-1}}{\hat{\sigma}} - \frac{e^{t-2}}{\hat{\sigma}} & 0 \end{bmatrix},$$

$$\varphi(\sigma,t) = \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ -\frac{\sigma(\sigma^2-1)(e^{-t}-e^{t})}{2\hat{\sigma}} & \frac{\sigma(e^{-t}-e^{t})}{2\hat{\sigma}} & e^t - \frac{te^t(\sigma^2e^{-1}+\sigma^2e^{-2}-e^{-2})}{\hat{\sigma}} & 0 \\ -\frac{\sigma(e^{-t}-e^{t})}{2\hat{\sigma}} & -\frac{\sigma(\sigma^2-2)(e^{-t}-e^{t})}{2\hat{\sigma}} & -\frac{te^t(\sigma^2e^{-1}-e^{-1}+e^{-2})}{\hat{\sigma}} & e^t \end{bmatrix},$$

$$\phi_u(t) = \begin{bmatrix} \frac{e^{-t}}{\sigma^2-1} & 0 & \frac{\sigma e^{t-1}}{\sigma^2-1} & 0 \\ 0 & 0 & \frac{1}{\sigma}e^{t-2} & 0 \end{bmatrix},$$

$$\Phi = \frac{1}{\hat{\sigma}} \begin{bmatrix} \sigma^2 e^{-t} & (\sigma^2-1)e^{-t} & \sigma e^{t-1}(\sigma^2-1+e^{-1}) & 0 \\ (\sigma^2-1)e^{-t} & e^{-t} & \sigma e^{t-1}(1+(\sigma^2-2)e^{-1}) & 0 \end{bmatrix},$$

where

$$\hat{\sigma} = \sigma^4 - 3\sigma^2 + 1$$

then the Hankel singular values, from Theorem 5.3, $\sigma$ are determined by

$$\det\left(\begin{bmatrix} e^{-1} & -1 & 0 & 0 \\ -\frac{\sigma(\sigma^2-1)(e^1-e^{-1})}{2\hat{\sigma}} & -\frac{\sigma(e^1-e^{-1})}{2\hat{\sigma}} & e - \frac{e(\sigma^2e^{-1}+\sigma^2e^{-2}-e^{-2})}{\hat{\sigma}} & -1 \\ -\frac{\sigma(e^1-e^{-1})}{2\hat{\sigma}} & -\frac{e^{-1}}{2\sigma} - \frac{(\sigma^2-2)(e^1-e^{-1})}{2\sigma\hat{\sigma}} & -\frac{e^1(\sigma^2e^{-1}-e^{-1}+e^{-2})}{\hat{\sigma}} & e \\ -\frac{\sigma(e^1-e^{-1})}{2\hat{\sigma}} & -\frac{e^{-1}(\sigma^2-1+e^{-1})}{\hat{\sigma}} & -\frac{0.0092}{\sigma} - \frac{\sigma e^{-4}(\sigma^2+\sigma^2e^2-e^2+2e^1-2)(e^2-1)}{2\hat{\sigma}} & 0 \end{bmatrix}\right) = 0$$

The possible solutions are 0.074903, 0.602657, 0.619083, 1.587781, and 2.063865. Thus the Hankel norm of this system is 2.063865.

**Case 2** : $b_1 = b_2 = 1$, i.e. the system realization is given by:

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) + u(t-1) + u(t-2) \\ y(t) &= x(t) + u(t-1) + u(t-2) \end{aligned}$$

where the effect of $B_j$ is included in. Similarly the Hankel norm by Lemma 5.2 is 5. By using our MATLAB code, Hankel norm of this system is 3.33544 which is larger than the previous case with $B_j$ terms.

# Chapter 6

# Conclusions

This thesis constructs the computational method for the Hankel norm of stable linear time-invariant systems with multiple input delays. Some conclusions and discussions could be drawn as follows :

## 6.1 Conclusion and Discussion

1. The stability of delay systems is analyzed by using the concept of stability equivalence. Under the existence of the solution of certain matrix algebraic equation, the stability of delay systems is determined by the system matrix of a delay-free system.

2. The Hankel operator for linear systems with feedthrough-type input delays and its adjoint are constructed.

3. The compactness of this operator is then examined. In general the Hankel operator of these systems are noncompact unless there is no feedthrough terms in input delays.

4. The norm computation of Hankel operator is studied in two different approaches :

   (a) Based on definition of induced operator norm which is transformed into one-parameter optimization problem and then solved by using variational principle.

   (b) Based on the fact that the value of norm is equal to the largest singular value of the operator and we arrive at a set of integral equations.

After change of variables and using lifting technique, the key issues in these two approaches are resolved into solving a system of differential-algebraic equations defined on the closed interval with delay-time as its length.

5. The Hankel norm is characterized as largest zero of the determinant of a complex matrix containing the value of norm (or equivalently, singular value) its unknown variable. The effect of delay time-length and the influence of the number of delays are included in the entries of this complex matrix.

6. Some illustrative examples are presented to demonstrate Hankel norm computation which shows that although the magnitude of the Hankel norm will increases when the time-length and number of delays increase but there exists an upper bound for large delay time.

## 6.2 Future Direction

The following problem are worthy for future study :

1. How to find $\Theta(t)$ in chapter 3 ?

2. How to construct the Hankel operator and adjoint Hankel operator for dynamical systems with state delay ?

3. How to compute the Hankel norm and Hankel singular values for state-delay systems ?

4. Model reduction : given an $m \times m$ transfer function $G \in \mathcal{RH}^\infty$ with McMillan degree $n$, how to find an $m \times m$ transfer function $\hat{G} \in \mathcal{RH}^\infty$ with McMillan degree no more than $k$ and $k < n$, such that $\|G - \hat{G}\|_H$ is minimized ?

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# Appendix A

# Numerical Codes

## A.1 Matlab code for solving $e^{[A+\Theta(0)]^T}\Theta(0) = A_1$

```
%object: compute solution of g
%use Newton method
%g=Q-expm(-A-Q)*B where Q,A,B are matrix
%If g is a matrix
%  f:M(R)_2x2 ----->M(R)_2x2
%    [a b]              [g(1) g(2)]
%    [c d]   |--------> [g(3) g(4)]
%If g is a map
%  g:R^4 ---> R^4
%   (a,b,c,d) |----->(g(1),g(2),g(3),g(4))
% use Numerical Differentiation compute D(g)_4x4

clear format long global A; global B;
%A=[-1 -3;2 -5];
%B=[1.66 -0.697;0.93 -0.33];
A=[-1 0;0 -2]; B=[1 0;0 0.5]; double(A); double(B);
%---------------initial data---------------------------------
Q=input('Enter inital Q(0)='); error=input('Enter error=');
t=cputime; x(1)=Q(1,1); x(2)=Q(1,2); x(3)=Q(2,1); x(4)=Q(2,2);
double(Q); g=double(Q-expm(-A-Q)*B); x=x'; S=x'; y=x+1; N=0;
```

```
J=Jcobin4(x); bar= waitbar(0,'Please wait...'); flag=1;
%---------------------newton method--------------------------
while (norm(g,inf)>error | norm(y-x,inf)>error) & N<100
   if abs(det(J))<10^(-4)
      close(bar);
      disp('input other Q');
      exercise %recurive(file name is exercise)
      flag=0;
      return %break out while loop
    else
       g=double(Q-expm(-A-Q)*B);
       J=Jcobin4(x);
       y=x;
       x=x-inv(J)*[g(1);g(2);g(3);g(4)];
       Q(1,1)=x(1);
       Q(1,2)=x(2);
       Q(2,1)=x(3);
       Q(2,2)=x(4);
       S=[S;x'];
       N=N+1;
        waitbar(N/100,bar);
   end
end

close(bar);
%----------------output data------------------------------------------
if N<100 & flag==1
  disp(' result:');
   fprintf(' iteration number=%d\n',N);
   fprintf(' run time=%f sec\n',cputime-t);
  disp(' Error =');
  disp(g);
  disp('  roots are=');
  disp('N  |                    X                  |');
```

```
  disp('-------------------------------------------');
  for i=1:N
    fprintf('%d  |',i);
    disp(S(i,1:4));
  end
else
   disp('maximum number of iterations exceeded');
end




% object:compute Jcobin of g
%
% use Numerical Differentiation
%
% f'(a)=(f(a+h)-f(a-h))/2h + O(h^2)
%
%   g_negative_h(1) mean g1(a-h)
%   g_plus_h(1)     mean g2(a+h)
%
% So g1'(a)=(g1(a+h)-g(a-h))/2h + O(h^2)
function J=Jcobin4(x)

global A; global B;
%A=[-1 -3;2 -5];
%B=[1.66 -0.697;0.93 -0.33];

Q=[x(1) x(2);x(3) x(4)];

double(Q);

h=10^(-5);

double(x);
```

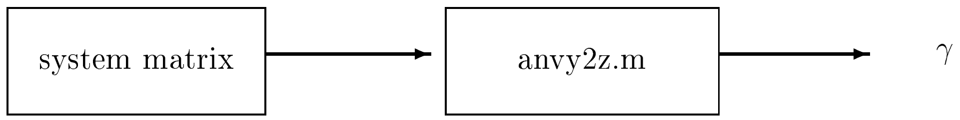
```
temp=x;

for i=1:4
   for j=1:4

        x(j)=x(j)+h;

        Q=[x(1) x(2);x(3) x(4)];

        g_plus_h=double(Q-expm(-A-Q)*B);

        x=temp;

        x(j)=x(j)-h;

        Q=[x(1) x(2);x(3) x(4)];

        g_negative_h=double(Q-expm(-A-Q)*B);

        J(i,j)=double((g_plus_h(i)-g_negative_h(i))/(2*h));

        x=temp;
    end
end
```

## A.2 Matlab code for computing Hankel norm

Matlab code for computing Hankel norm by Theorem 5.2.

**Case 1**:

If $b_1$ and $b_2$ terms are zero, then we can directly solve the solutions



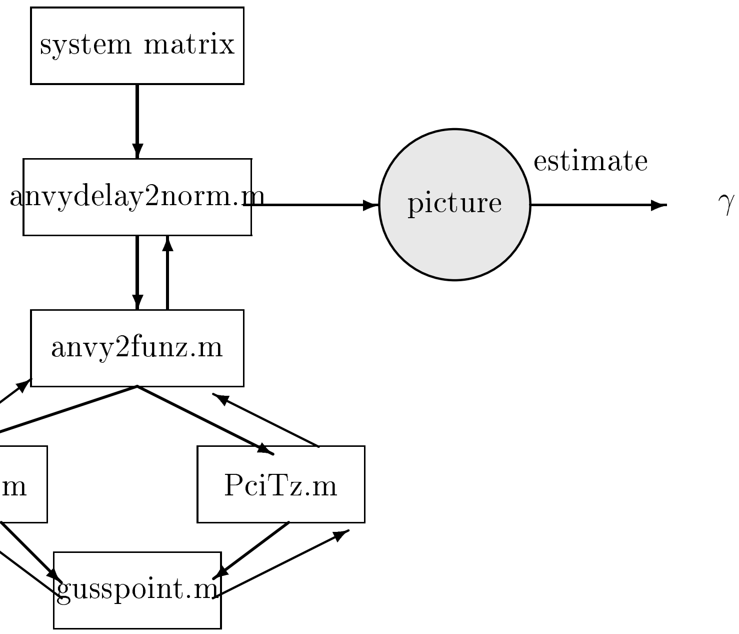

```
%anvymatrix2.m
% function svd=anvy(A,B0,B1,B2,C,D1,D2,T)
% define symbolic variables for further use
global A B0 B1 B2 C D1 D2 T m n p q N x c

syms s t k ; % s=sym(’s’);t=sym(’t’);k=sym(’k’);

% Declare the system parameters
% A=sym(’-1’);B0=sym(’1’);B1=sym(’0’);B2=sym(’0’);C=sym(’1’);
%D1=sym(’0’);D2=sym(’0’);
% T=sym(’1’);
A=-1; B0=1; B1=0; B2=0; C=1; D1=1; D2=0; T=2;
q=2; n=1; m=1; p=1; N=2;

D=[D1 D2;D2 zeros(p,m)];
Ds=s^2*eye(q)-D’*D;

Iq=eye(q);   % Iq=sym(’[1 0;0 1]’);

At=kron(Iq,A);

Ct=kron(Iq,C’*C); Ahat=[At zeros(q*n);-(1/s)*Ct (-1)*At’]+[B1
B2;B2 zeros(n,m);-(1/s)*C’*D1 -(1/s)*C’*D2;-(1/s)*C’*D2
zeros(n,m)]*inv(Ds)*[D1’*C D2’*C s*B1’ s*B2’;D2’*C zeros(m,n)
```

```
s*B2' zeros(n,m)]

Bhat=[B1 B2;B2 zeros(n,m);-(1/s)*C'*D1 -(1/s)*C'*D2;-(1/s)*C'*D2
zeros(n,m)]*inv(Ds)*[0 s*B0'*expm(-A'*(k-T));0
s*(B0'*expm(-A'*(k-2*T))+B1'*expm(-A'*(k-T)))];

Chat=expm(Ahat*t)*[1 0;0 0;0 1;0 0]+int(expm(Ahat*(t-k))*Bhat,k,0,t)
ChatT=subs(Chat,'t',T);
%ChatT=Chatfun(s,T);

Dhat=expm(Ahat*t)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0]
DhatT=subs(Dhat,'t',T);

Wt=Chat+Dhat*inv(eye(2*q)-DhatT)*ChatT

WT=subs(Wt,'t',T);

Ht=inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n) s*B2' zeros(n,m)]*...
Wt+inv(Ds)*[zeros(m,n) s*B0'*expm(-A'*(t-T));zeros(m,n)
s*(B0'*expm(-A'*(t-2*T))+B1'*expm(-A'*(t-T)))]

%Q=int(expm(A'*t)*C'*C*expm(A*t),t,0,inf)
%Pn=int(expm(-A*t)*(B0+expm(-A*T)*B1+expm(-A*2*T)*B2)*(B0'+B1'*expm(-A'*T)...
%+B2'*expm(-A*2*T))*expm(-A'*t),t,-inf,0)
sys=ss(A,B0,C,zeros(p,m));
Q=gram(sys,'o');

sysn=ss(A,expm(-2*T*A)*B2+expm(-T*A)*B1+B0,C,zeros(p,m));
Pn=gram(sysn,'c');

G1=int(expm(-A*(t-2*T))*B0*Ht(2,1)+
expm(-A*(t-T))*(B0*Ht(1,1)+B1*Ht(2,1)),t,0,T);
%G1=PsiT(s,T)
G2=int(expm(-A*(t-2*T))*B0*Ht(2,2)+
```

```
expm(-A*(t-T))*(B0*Ht(1,2)+B1*Ht(2,2)),t,0,T);


P=expm(N*A*T)*Pn*expm(N*A'*T); format long;

F=[(1/s)*Q -1;0 0]*[WT(q,1) WT(q,2);WT(2*q,1) WT(2*q,2)]+[0
0;-1+G1 (1/s)*P+G2];
detF=expand(det(F))
svd=double(solve(vpa(detF),s))
```

**Case 2**:

If $b_1$ and $b_2$ terms are not both zero, then the process of solving is




```
global A B0 B1 B2 C D1 D2 T m n p q N x c

A=-1; B0=1; B1=0; B2=0; C=1; D1=0; D2=0; T=1;

q=2; n=1; m=1; p=1; N=2;
```

```
format rat; D=[D1 D2;D2 zeros(p,m)];
spect_radius=sqrt(max(eig(D'*D))) hold on;

      begin_sigma=spect_radius+0.1;

N=20; length_sigma=1.5;
end_sigma=begin_sigma+length_sigma;
delta_sigma=length_sigma/N;

values=zeros(N+1,2);

for i=1:N+1
   sigma=begin_sigma+delta_sigma*(i-1);
   values(i,1)=sigma
   values(i,2)=anvy2funz(sigma)

end
plot(values(:,1),values(:,2)) axis([0.1,2,-10,10]);
grid on

%anvy2funz.m
function value=anvy2funz(sigma)
global A B0 B1 B2 C D1 D2 T m n p q N x c
gusspoint(10);
s=sigma; syms t k;
D=[D1 D2;D2 zeros(p,m)];

Ds=s^2*eye(q)-D'*D;

Iq=eye(q);   % Iq=sym('[1 0;0 1]');

At=kron(Iq,A);

Ct=kron(Iq,C'*C);
```

```
Ahat=[At zeros(q*n);-(1/s)*Ct (-1)*At']+[B1 B2;B2
zeros(n,m);-(1/s)*C'*D1 -(1/s)*C'*D2;-(1/s)*C'*D2
zeros(n,m)]*inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n)
s*B2' zeros(n,m)];


%ChatT=Chatfun(s,T);

%Dhat=expm(Ahat*t)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0];

DhatT=expm(Ahat*T)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0];


WT=Chatfunz(s,T)+DhatT*inv(eye(2*q)-DhatT)*Chatfunz(s,T);

%WT=subs(Wt,'t',T);

%Q=int(expm(A'*t)*C'*C*expm(A*t),t,0,inf)
%Pn=int(expm(-A*t)*(B0+expm(-A*T)*B1+expm(-A*2*T)*B2)*(B0'+B1'*expm(-A'*T)...
%+B2'*expm(-A*2*T))*expm(-A'*t),t,-inf,0);
sys=ss(A,B0,C,zeros(p,m)); Q=gram(sys,'o');

sysn=ss(A,expm(-2*T*A)*B2+expm(-T*A)*B1+B0,C,zeros(p,m));
Pn=gram(sysn,'c');

G1=PsiTz(s,T);

G2=PciTz(s,T);

P=expm(N*A*T)*Pn*expm(N*A'*T);

F=[Q 1;0 0]*[WT(q,1) WT(q,2);WT(2*q,1) WT(2*q,2)]+[0 0;1-G1
(1/s^2)*P-G2];
```

```
detF=det(F);

format long ;

value=real(det(vpa(F)));

%chatfunz.m
function value=Chatfun(s,t)
global A B0 B1 B2 C D1 D2 T m n p q N x c



D=[D1 D2;D2 zeros(p,m)]; Ds=s^2*eye(q)-D'*D;

Iq=eye(q);   % Iq=sym('[1 0;0 1]');

At=kron(Iq,A);

Ct=kron(Iq,C'*C); Ahat=[At zeros(q*n);-(1/s)*Ct (-1)*At']+[B1
B2;B2 zeros(n,m);-(1/s)*C'*D1 -(1/s)*C'*D2;-(1/s)*C'*D2
zeros(n,m)]*inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n)
s*B2' zeros(n,m)];
% Initialize

   a = 0;
   b = t;
   if nargin<3
      np = 10;
      end


  y = 0;
  % Estimate integral
```

```
alpha = (b - a)/2;
 beta = (b + a)/2;
 for i = 1 : np
 k = alpha*x(i)+beta;

      expmkTA  = expm(-(k-T)*A');
      expmk2TA = expm(-(k-2*T)*A');

Bhat=[B1 B2;B2 zeros(n,m);-(1/s)*C'*D1 -(1/s)*C'*D2;-(1/s)*C'*D2
zeros(n,m)]*inv(Ds)*[0 s*B0'*expm(-A'*(k-T));0
s*(B0'*expm(-A'*(k-2*T))+B1'*expm(-A'*(k-T)))];

             y = y + c(i)*expm(-Ahat*k)*Bhat;
     end
   y = alpha*y;
%
%********************************************************************************
%

value=expm(t*Ahat)*([1 0;0 0;0 1;0 0]+y);

%PicTz.m
function value=PciT(s,k) global A B0 B1 B2 C D1 D2 T m n p q N x c

D=[D1 D2;D2 zeros(p,m)]; Ds=s^2*eye(q)-D'*D;

Iq=eye(q);   % Iq=sym('[1 0;0 1]');

At=kron(Iq,A);

Ct=kron(Iq,C'*C);

Ahat=[At zeros(q*n);-(1/s)*Ct (-1)*At']+[B1 B2;B2
```

```
zeros(n,m);-(1/s)*C'*D1 -(1/s)*C'*D2;-(1/s)*C'*D2
zeros(n,m)]*inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n)
s*B2' zeros(n,m)];

ChatT=Chatfunz(s,T);


% Initialize


   a = 0;
   b = k;
      np = 10;

   gusspoint(10);
    y = 0;


   % Estimate integral

         alpha = (b - a)/2;
         beta  = (b + a)/2;
   for i = 1 : np

      k=alpha*x(i)+beta;

      Dhat=expm(Ahat*k)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0];

      DhatT=expm(Ahat*T)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0];

      Wk=Chatfunz(s,k)+Dhat*inv(eye(2*q)-DhatT)*ChatT;

      Hk=inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n) s*B2' zeros(n,m)]*...
       Wk+inv(Ds)*[zeros(m,n) s*B0'*expm(-A'*(k-T));zeros(m,n)
```

```
                   s*(B0'*expm(-A'*(k-2*T))+B1'*expm(-A'*(k-T)))];

     Gk=expm(-A*(k-2*T))*B0*Hk(2,2)+expm(-A*(k-T))*(B0*Hk(1,2)+B1*Hk(2,2));

     y =y+c(i)*Gk;
   end
   y = alpha*y;

%********************************************************************************
%

value=y;

%PsiTz.m
function value=PsiT(s,k) global A B0 B1 B2 C D1 D2 T m n p q N x c


D=[D1 D2;D2 zeros(p,m)]; Ds=s^2*eye(q)-D'*D;

Iq=eye(q);   % Iq=sym('[1 0;0 1]');

At=kron(Iq,A);

Ct=kron(Iq,C'*C);

Ahat=[At zeros(q*n);-(1/s)*Ct (-1)*At']+[B1 B2;B2
zeros(n,m);-(1/s)*C'*D1 -(1/s)*C'*D2;-(1/s)*C'*D2
zeros(n,m)]*inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n)
s*B2' zeros(n,m)];

ChatT=Chatfunz(s,T);
```

```
% Initialize


   a = 0;
   b = k;

      np = 10;
   gusspoint(10);

   y = 0;

      % Estimate integral

         alpha = (b - a)/2;
         beta  = (b + a)/2;
   for i = 1 : np

      k=alpha*x(i)+beta;

      Dhat=expm(Ahat*k)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0];

      DhatT=expm(Ahat*T)*[0 0 0 0;1 0 0 0;0 0 0 0;0 0 1 0];

      Wk=Chatfunz(s,k)+Dhat*inv(eye(2*q)-DhatT)*ChatT;

      Hk=inv(Ds)*[D1'*C D2'*C s*B1' s*B2';D2'*C zeros(m,n) s*B2' zeros(n,m)]*...
         Wk+inv(Ds)*[zeros(m,n) s*B0'*expm(-A'*(k-T));zeros(m,n)
                     s*(B0'*expm(-A'*(k-2*T))+B1'*expm(-A'*(k-T)))];

     Gk=expm(-A*(k-2*T))*B0*Hk(2,1)+expm(-A*(k-T))*(B0*Hk(1,1)+B1*Hk(2,1));

     y =y+c(i)*Gk;
   end
```

```
   y = alpha*y;

%*********************************************************************************
%

value=y;

function  gusspoint(np) global A B0 B1 B2 C D1 D2 T m n p q N x c



   x = zeros (np,1);
   c = zeros (np,1);

% Compute parameters

   switch (np)

      case 1; x(1) = 0.0;
              c(1) = 2.0;

      case 2; x(1) = 0.5773503; x(2) = -x(1);
              c(1) = 1; c(2) = c(1);

      case 3; x(1) = 0.0;
              x(2) = 0.7745967; x(3) = -x(2);
              c(1) = 0.8888889;
              c(2) = 0.5555556; c(3) = c(2);

      case 4; x(1) = 0.3399810; x(2) = -x(1);
              x(3) = 0.8611363; x(4) = -x(3);
              c(1) = 0.6521452; c(2) = c(1);
              c(3) = 0.3478548; c(4) = c(3);

      case 5; x(1) = 0.0;
```

```
        x(2) = 0.5384693; x(3) = -x(2);
        x(4) = 0.9061798; x(5) = -x(4);
        c(1) = 0.5688880;
        c(2) = 0.4786287; c(3) = c(2);
        c(4) = 0.2369269; c(5) = c(4);

case 6; x(1) = 0.2386192; x(2) = -x(1);
        x(3) = 0.6612094; x(4) = -x(3);
        x(5) = 0.9324695; x(6) = -x(5);
        c(1) = 0.4679139; c(2) = c(1);
        c(3) = 0.3607616; c(4) = c(3);
        c(5) = 0.1713245; c(6) = c(5);

case 7; x(1) = 0.0;
        x(2) = 0.4058452; x(3) = -x(2);
        x(4) = 0.7415312; x(5) = -x(4);
        x(6) = 0.9491079; x(7) = -x(6);
        c(1) = 0.4179592;
        c(2) = 0.3818301; c(3) = c(2);
        c(4) = 0.2797054; c(5) = c(4);
        c(6) = 0.1294850; c(7) = c(6);

case 8; x(1) = 0.1834346; x(2) = -x(1);
        x(3) = 0.5255324; x(4) = -x(3);
        x(5) = 0.7966665; x(6) = -x(5);
        x(7) = 0.9620899; x(8) = -x(7);
        c(1) = 0.3626838; c(2) = c(1);
        c(3) = 0.3137066; c(4) = c(3);
        c(5) = 0.2223810; c(6) = c(5);
        c(7) = 0.1012285; c(8) = c(7);

case 9; x(1) = 0.0;
        x(2) = 0.3242534; x(3) = -x(2);
        x(4) = 0.6133714; x(5) = -x(4);
```

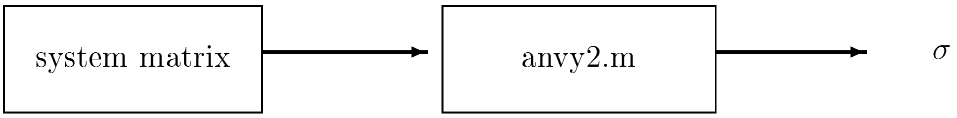
```
        x(6) = 0.8360311; x(7) = -x(6);
        x(8) = 0.9681602; x(9) = -x(8);
        c(1) = 0.3302394;
        c(2) = 0.3123471; c(3) = c(2);
        c(4) = 0.2606107; c(5) = c(4);
        c(6) = 0.1806482; c(7) = c(6);
        c(8) = 0.0812744; c(9) = c(8);

case 10; x(1) = 0.1488743; x(2) = -x(1);
         x(3) = 0.4333954; x(4) = -x(3);
         x(5) = 0.6794096; x(6) = -x(5);
         x(7) = 0.8650634; x(8) = -x(7);
         x(9) = 0.9739065; x(10) = -x(9);
         c(1) = 0.2955242; c(2) = c(1);
         c(3) = 0.2692602; c(4) = c(3);
         c(5) = 0.2190864; c(6) = c(5);
         c(7) = 0.1494513; c(8) = c(7);
         c(9) = 0.0666713; c(10) = c(9);
end
```

## A.3 Matlab code for computing Hankel singular value

Matlab code for computing Hankel singular value by Theorem 5.3.

**Case 1**:

If $b_1$ and $b_2$ terms are zero, then we can directly solve the solutions

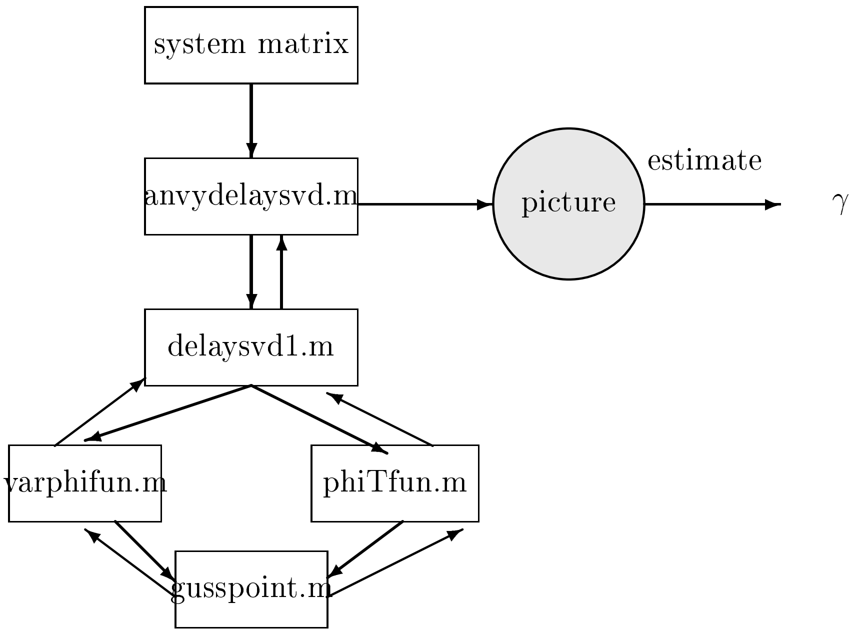



```
global A B0 B1 B2 C D1 D2 T m n p q

A=-1; B0=0; B1=1; B2=0; C=1; D1=0; D2=0; T=0;

q=2; n=1; m=1; p=1;

syms s t k ; % s=sym('s');t=sym('t');k=sym('k');


% n - dimensions of the state-space
% m - no. of input varibles
% n - no. of output varibles
% q - no. of delays
Iq=eye(q);           % Iq    = identity matrix of size q x q;


varphit=varphifun(s,t);
%varphi(s,t)=expm(t*As)+int(expm((t-tau)*As)*Bs,tau,0,t);
varphiT=varphifun(s,T);
%varphi(s,T)=expm(T*As)+int(expm((T-tau)*As)*Bs,tau,0,T);


sys=ss(A,B0,C,zeros(p,m)); Q=gram(sys,'o');
%
```

```
%Pn=int(expm((-1)*k*A)*(expm(2*T*A)*B0+expm(T*A)*B1+B2)*...
%    (B0'*expm(2*T*A')+B1'*expm(T*A')+B2')*expm((-1)*k*A'),k,-inf,0);
%expmtA=expm(t*A);
sysn=ss(A,expm(2*T*A)*B0+expm(T*A)*B1+B2,C,zeros(p,m));
Pn=gram(sysn,'c');

%PHIt=int([zeros(n,m) zeros(n,m) subs(expmtA,'t',(-1)*(t-T))*B0...
%subs(expmtA,'t',(-1)*(t-2*T))*B0+subs(expmtA,'t',(-1)*(t-T))*B1]*Ht,t,0,T);
%PHIt=simplify(PHIt);
PhiT=phiTfun(s,T);

R=zeros(2*q,2*q);%R=[In(1,:);In(3,:);In(2,:);In(4,:)];
%R=[In(1,:);In(3,:);In(2,:);In(4,:)];
for i=1:q
   j=2*i-1;
   R(j,i)=R(j,i)+1;
   R(j+1,i+q)=R(j+1,i+q)+1;
end
F=[1 0 0 0;0 1 0 0;0 0 -1/s*Q 1;0 0 0 0]*R*varphiT+...
[0 0 -1 0;0 0 0 -1;0 0 0 0;(1-PhiT(:,1)) -(1/s*Pn+PhiT(:,3)) -PhiT(:,2) -PhiT(:,4)]...
*R;
detF=expand(det(F));%
format long ; %
svd=solve(vpa(detF),s)
```

**Case 2**:

If $b_1$ and $b_2$ terms are not both zero, then the process of solving is




```
%anvydelaysvd.m
clear global A B0 B1 B2 C D1 D2 T m n p q x c

A=-1; B0=0; B1=1; B2=0; C=1; D1=1; D2=0;
%T=3;

q=2; n=1; m=1; p=1; format rat; D=[D1 D2;D2 zeros(p,m)];
spect_radius=sqrt(max(eig(D'*D))) T=1;
%for qq=1:20
   for j=0:100
   begin_sigma=spect_radius+0.1*j;

   hold on
N=40; length_sigma=0.1; end_sigma=begin_sigma+length_sigma;
delta_sigma=length_sigma/N;
```

```
values=zeros(N+1,2);

for i=1:N+1
   sigma=begin_sigma+delta_sigma*(i-1);
   values(i,1)=sigma
   values(i,2)=delaysvd1(sigma)

end plot(values(:,1),values(:,2)) axis([1,2.5,-10,10]); grid on

end
%T=T+0.2
%end

%delaysvd1.m
function value=delaysvd1(sigma)
global A B0 B1 B2 C D1 D2 T m n p q x c

s=sigma;
syms t k;

Iq=eye(q);   % Iq=sym('[1 0;0 1]');
gusspoint(10);
varphiT=varphifun(s,T);
%varphi(s,T)=expm(T*As)+int(expm((T-tau)*As)*Bs,tau,0,T);

% Q=int(expm(k*A)*C'*C*expm(k*A),k,0,inf);
sys=ss(A,B0,C,zeros(p,m)); Q=gram(sys,'o');

%Pn=int(expm((-1)*k*A)*(expm(2*T*A)*B0+expm(T*A)*B1+B2)*...
%   (B0'*expm(2*T*A')+B1'*expm(T*A')+B2')*expm((-1)*k*A'),k,-inf,0);
expmtA=expm(t*A);
sysn=ss(A,expm(2*T*A)*B0+expm(T*A)*B1+B2,C,zeros(p,m));
Pn=gram(sysn,'c');
```

```
PhiT=phiTfun(s,T);
%R=[1 0 0 0;0 0 1 0;0 1 0 0;0 0 0 1];
R=zeros(2*q*n,2*q*n); for i=1:q*n
   j=2*i-1;
   R(j,i)=R(j,i)+1;
   R(j+1,i+q*n)=R(j+1,i+q*n)+1;
end

%F=[1 0 0 0;0 1 0 0;0 0 -1/s*Q 1;0 0 0 0]*R*WT+
%[0 0 -1 0;0 0 0 -1;0 0 0 0;(1-PHIt(1,1)) -(1/s*Pn+PHIt(1,3))
%-PHIt(1,2) -PHIt(1,4)]*R;
F=[eye(2*(q-1)*n) zeros(2*(q-1)*n,2*n);zeros(2*n,2*(q-1)*n)
[-1/s*Q eye(n);zeros(n) zeros(n)]]*R*varphiT...
+[zeros(2*(q-1)*n,2*n)
(-1)*eye(2*(q-1)*n);zeros(n,2*q*n);eye(n)+(-1)*PhiT(:,1:n)
-(1/s*Pn+PhiT(:,2*n+1:3*n)) -PhiT(:,n+1:2*n)-PhiT(:,3*n+1:4*n)]*R;

value=det(vpa(F));

%varphifun.m
function value=varphifun(s,t)

%*******************************************************************
% Usage:  value=varphifun(s,t)
%
% Purpose: compute the following function
%
%       \varphi(\sigma,t) = e^{t*A_{\sigma}}
%          +\int_{0}^{t} e^{A_{\sigma}(t-tau)} B_{\sigma}(\tau) d\tau;
%
%********************************************************************
%
% Mnemonics:
%
```

```
%    Inputs:          s        - denote the singular value \sigma
%                     t        - time
%
%    local variables: Iq       - identity matrix of size q x q;
%                    %
%    Outputs:         value    - the value of function varphi at (s,t)
%
%    Global variables:
%       A        - the system state matrix A
%       B0       - the system input matrix B0 for non-delay terms
%       B        - collect all the system input matrix related to delay in
%                  state equation, i.e., B=[B_1 B_2 ... B_{q-1} B_q]
%                                          [B_2 B_3 ... B_q     0  ]
%                                          [:    :       :      :  ]
%                                          [B_q  0  ...  0      0  ]
%       C        - the system output matrix
%       D        - collect all the system input matrix related to delay in
%                  output equation, i.e., D=[D_1 D_2 ... D_{q-1} D_q]
%                                           [D_2 D_3 ... D_q     0  ]
%                                           [:    :       :      :  ]
%                                           [D_q  0  ...  0      0  ]
%       T        - unit delay time
%       m        - the dimension of input vector
%       n        - the dimension of state vector
%       p        - the dimension of output vector
%       q        - the number of delays
%
%*****************************************************************************
%

global A B0 B1 B2 C D1 D2 T m n p q x c


Iq=eye(q);           % Iq    = identity matrix of size q x q;
```

```
D=[D1 D2;D2 zeros(p,m)];

At=kron(Iq,A); B=[B1 B2;B2 zeros(n,m)]; Ct=kron(Iq,C);
Ds=s*eye(2*(p+m))-[zeros(2*p,2*p) D;D' zeros(2*m,2*m)] As=[At
zeros(2*n,2*n);zeros(2*n,2*n) (-1)*At']+...
[zeros(2*n,2*p) B;(-1)*Ct' zeros(2*n,2*m)]*inv(Ds)*[Ct
zeros(2*p,2*n);zeros(2*m,2*n) B']
%***************************************************************************
%
% Description:
%   The following code use the n-point Gauss-Legendre formula to numerically
%   integrate the function  e^{A_{\sigma}(t-tau)} B_{\sigma}(\tau)
%   over the interval [a,b].
%
% Inputs:      a  = lower limit of integration
%              b  = upper limit of integration
%              np = number of points (1 <= np <= 10)
%
% Outputs:     y = estimate of integral
%
%****************************************************************************
%
% Initialize

   a = 0;
   b = t;
   if nargin<3
      np = 10;
   end
   y = 0;
    gusspoint(10);
% Estimate integral
```

```
   alpha = (b - a)/2;
   beta  = (b + a)/2;
   for i = 1 : np
      k = alpha*x(i)+beta;
     expmkTA  = expm(-(k-T)*A');
     expmk2TA = expm(-(k-2*T)*A');
Bk=[B0'*expmkTA zeros(m,n);B0'*expmk2TA+B1'*expmkTA zeros(m,n)];
Bs=[zeros(2*n,2*p) B;(-1)*Ct' zeros(2*n,2*m)]*inv(Ds)*...
[zeros(2*p,2*n) zeros(2*p,2*n);zeros(2*m,2*n) Bk]

          y = y + c(i)*expm(-As*k)*Bs;
   end
   y = alpha*y;
%
%***************************************************************************

value=expm(t*As)*(eye(size(As))+y);

%phiTfun.m
function value=phiTfun(s,t)
%
%***************************************************************************
% Usage:  value=varphifun(s,t)
%
% Purpose: compute the following function
%        \varphi(\sigma,T) = e^{T*A_{\sigma}}
%              +\int_{0}^{T} e^{A_{\sigma}(T-tau)} B_{\sigma}(\tau) d\tau;
%
%***************************************************************************
%
% Mnemonics:
%
%    Inputs:          s        - denote the singular value \sigma
%                     t        - time
```

```
%
%    local variables: Iq       - identity matrix of size q x q;
%                     %
%    Outputs:          value    - the value of function varphi at (s,t)
%
%    Global variables:
%         A        - the system state matrix A
%         B0       - the system input matrix B0 for non-delay terms
%         B        - collect all the system input matrix related to delay in
%                    state equation, i.e., B=[B_1 B_2 ... B_{q-1} B_q]
%                                            [B_2 B_3 ... B_q     0  ]
%                                            [:    :       :      :  ]
%                                            [B_q  0  ...  0      0  ]
%         C        - the system output matrix
%         D        - collect all the system input matrix related to delay in
%                    output equation, i.e., D=[D_1 D_2 ... D_{q-1} D_q]
%                                             [D_2 D_3 ... D_q     0  ]
%                                             [:    :       :      :  ]
%                                             [D_q  0  ...  0      0  ]
%         T        - unit delay time
%         m        - the dimension of input vector
%         n        - the dimension of state vector
%         p        - the dimension of output vector
%         q        - the number of delays
%
%****************************************************************************
%


global A B0 B1 B2 C D1 D2 T m n p q x c

Iq=eye(q);            % Iq    = identity matrix of size q x q;

At=kron(Iq,A); B=[B1 B2;B2 zeros(n,m)]; Ct=kron(Iq,C); D=[D1 D2;D2
```

```
zeros(p,m)]; Ds=s*eye(2*(p+m))-[zeros(q*p,q*p) D;D'
zeros(q*m,q*m)];

As=[At zeros(q*n,q*n);zeros(q*n,q*n) (-1)*At']+[zeros(q*n,q*p)
B;(-1)*Ct' zeros(q*n,q*m)]*inv(Ds)*...
   [Ct zeros(q*p,q*n);zeros(q*m,q*n) B'];

% Initialize

   a = 0;
   b = t;
   if nargin<3
      np = 10;
   end
   y = 0;
   gusspoint(10);
  % Estimate integral

   alpha = (b - a)/2;
   beta  = (b + a)/2;
   for i = 1 : np
      k = alpha*x(i)+beta;

       expmkTAT  = expm(-(k-T)*A');
       expmk2TAT = expm(-(k-2*T)*A');
       expmkTA  = expm(-(k-T)*A);
       expmk2TA = expm(-(k-2*T)*A);

     Bk=[B0'*expmkTAT zeros(m,n);B0'*expmk2TAT+B1'*expmkTAT zeros(m,n)];


varphit=varphifun(s,k);
%varphi(s,t)=expm(t*As)+int(expm((t-tau)*As)*Bs,tau,0,t);
```

```
Ht=inv(Ds)*([Ct zeros(q*p,q*n);zeros(q*m,q*n)
B']*varphit+[zeros(q*p,q*n) zeros(q*p,q*n);zeros(q*m,q*n) Bk]);

expmtA=expm(t*A); Phithat=[zeros(n,m) zeros(n,m)  expmkTA*B0
expmk2TA*B0+expmkTA*B1];

y = y + c(i)*Phithat*Ht;
   end
   y = alpha*y;

%*************************************************************************


value=y;


function  gusspoint(np)

global A B0 B1 B2 C D1 D2 T m n p q N x c

   x = zeros (np,1);
   c = zeros (np,1);

% Compute parameters

   switch (np)

      case 1; x(1) = 0.0;
              c(1) = 2.0;

      case 2; x(1) = 0.5773503; x(2) = -x(1);
              c(1) = 1; c(2) = c(1);

      case 3; x(1) = 0.0;
```

```
        x(2) = 0.7745967; x(3) = -x(2);
        c(1) = 0.8888889;
        c(2) = 0.5555556; c(3) = c(2);

case 4; x(1) = 0.3399810; x(2) = -x(1);
        x(3) = 0.8611363; x(4) = -x(3);
        c(1) = 0.6521452; c(2) = c(1);
        c(3) = 0.3478548; c(4) = c(3);

case 5; x(1) = 0.0;
        x(2) = 0.5384693; x(3) = -x(2);
        x(4) = 0.9061798; x(5) = -x(4);
        c(1) = 0.5688880;
        c(2) = 0.4786287; c(3) = c(2);
        c(4) = 0.2369269; c(5) = c(4);

case 6; x(1) = 0.2386192; x(2) = -x(1);
        x(3) = 0.6612094; x(4) = -x(3);
        x(5) = 0.9324695; x(6) = -x(5);
        c(1) = 0.4679139; c(2) = c(1);
        c(3) = 0.3607616; c(4) = c(3);
        c(5) = 0.1713245; c(6) = c(5);

case 7; x(1) = 0.0;
        x(2) = 0.4058452; x(3) = -x(2);
        x(4) = 0.7415312; x(5) = -x(4);
        x(6) = 0.9491079; x(7) = -x(6);
        c(1) = 0.4179592;
        c(2) = 0.3818301; c(3) = c(2);
        c(4) = 0.2797054; c(5) = c(4);
        c(6) = 0.1294850; c(7) = c(6);

case 8; x(1) = 0.1834346; x(2) = -x(1);
        x(3) = 0.5255324; x(4) = -x(3);
```

```
        x(5) = 0.7966665; x(6) = -x(5);
        x(7) = 0.9620899; x(8) = -x(7);
        c(1) = 0.3626838; c(2) = c(1);
        c(3) = 0.3137066; c(4) = c(3);
        c(5) = 0.2223810; c(6) = c(5);
        c(7) = 0.1012285; c(8) = c(7);

case 9; x(1) = 0.0;
        x(2) = 0.3242534; x(3) = -x(2);
        x(4) = 0.6133714; x(5) = -x(4);
        x(6) = 0.8360311; x(7) = -x(6);
        x(8) = 0.9681602; x(9) = -x(8);
        c(1) = 0.3302394;
        c(2) = 0.3123471; c(3) = c(2);
        c(4) = 0.2606107; c(5) = c(4);
        c(6) = 0.1806482; c(7) = c(6);
        c(8) = 0.0812744; c(9) = c(8);

case 10; x(1) = 0.1488743; x(2) = -x(1);
         x(3) = 0.4333954; x(4) = -x(3);
         x(5) = 0.6794096; x(6) = -x(5);
         x(7) = 0.8650634; x(8) = -x(7);
         x(9) = 0.9739065; x(10) = -x(9);
         c(1) = 0.2955242; c(2) = c(1);
         c(3) = 0.2692602; c(4) = c(3);
         c(5) = 0.2190864; c(6) = c(5);
         c(7) = 0.1494513; c(8) = c(7);
         c(9) = 0.0666713; c(10) = c(9);
end
```

# VITA

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**Publications :**

1. Huang-Nan Huang, Chiu-Chun Lin, and Fang-Bo Yeh, On computing singular values and vectors for a class of noncompact Hankel operators, Fifth SIAM Conference on Control & Its Applications, San Diego, July 11-14, 2001.
2. Huang-Nan Huang, Fang-Bo Yeh, and Chiu-Chun Lin, Hankel-norm computation for LTI systems with multiple feedthrough input delays, 40th IEEE Conference on Decision and Control, December, 2001. (accepted regular paper)