Abstract

Let Γ denote the set of symmetrized bidisc. In this thesis we discuss the Schwarz lemma on Γ also known as the special flat problem on Γ as:

Given $\alpha_2 \in \mathbb{D}$, $\alpha_2 \neq 0$ and $(s_2, p_2) \in \Gamma$, find an analytic function $\varphi : \mathbb{D} \to \Gamma$ with $\varphi(\lambda) = (s(\lambda), p(\lambda))$ satisfies

$$\varphi(0) = (0, 0),\ \varphi(\alpha_2) = (s_2, p_2)$$

Based on the equality of Carathéodory and Kobayashi distances, and the Schur's theorem, we construct an analytic function φ to solve this problem.

Keywords: Spectral Nevanlinna-Pick interpolation, Poincaré distance, Carathéodory distance, Kobayashi distance, Symmetrized bidisc, Schwarz lemma.

Contents

1 Introduction

1.1 Notations

Symbol	Meaning
$\mathbb{R}$	The real value
$\mathbb{C}$	The complex value
$\mathbb{D}$	The set of $\{\lambda : \lvert\lambda\rvert < 1\}$
$\overline{\mathbb{D}}$	The set of $\{\lambda : \lvert\lambda\rvert \leq 1\}$
$\mathbb{T}$	The unit circle
$\bar{\sigma}(A)$	The maximun singular value of A
$\rho(A)$, $\lVert \cdot \rVert_s$	The value of $\max\{\lvert\lambda\rvert : \lambda$ are the eigenvalues of A$\}$
Γ	The set of $\{(s,p) : \lambda^2 - s\lambda + p = 0,\ \lambda \in \mathbb{C},\ \lvert\lambda\rvert \leq 1\}$
$int(\Gamma)$	The set of $\{(s,p) : \lambda^2 - s\lambda + p = 0,\ \lambda \in \mathbb{C},\ \lvert\lambda\rvert < 1\}$
$T(M, \Delta)$	The transfer function of the closed loop system M and Δ
A_c	The companion matrix of A
$\mathbb{C}^{n\times n}$	The $n \times n$ matrices with complex elements
$\lVert \cdot \rVert_\infty$	The infinity norm
$\lVert \cdot \rVert_\mu$	The $\mu-$norm
$\partial\Omega$	The boundary in arbitrary set Ω
$C_\Omega(s,p)$	The Carathéodory distance between two points s and p in arbitrary set Ω

$K_\Omega(s,p)$	The Kobayashi distance between two points s and p in arbitrary set Ω
$d(s,p)$	The Poincaré distance between two points s and p in $\mathbb{D}$
$\mathrm{diag}[\lambda_1, ..., \lambda_n]$	The diagonal matrix with entries $\lambda_1, ..., \lambda_n$

1.2 Motivation

There are two kinds of uncertainties in control systems, structured uncertainty and unstructured uncertainty. H^∞-control theory solves the unstructured uncertainty systems by using the classical Nevanlinna-Pick interpolation theory. In robust H^∞-controller design problem, we consider the following configuration:

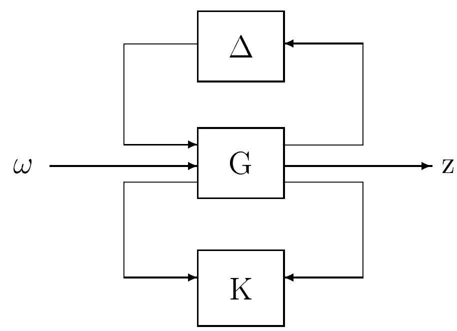

Figure 1: Control configuration for uncertain systems.

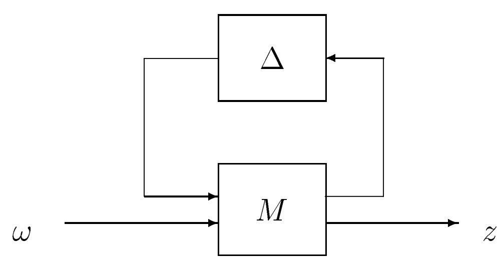

Figure 2: M denote the closed-loop system G+K.

Where G is a system with structured uncertainty Δ, and K is controller to be designed. Let M denote the closed-loop system transfer function $(G+K)$,i.e, and w and z are the noise and error signals, respectively. Two main essential ideas in robust H^{∞}-control theory are:

Robust Stability: $(M+\Delta)_{w=0,\ z=0}$ is internally stable.

Robust Performance: Let $T(M,\Delta)$ be the transfer function of the closed-loop system $(M+\Delta)$. It is required that

$$\sup_{w}\frac{\|z\|_2}{\|w\|_2} = \|T(M,\Delta)\|_{\infty} < r$$

for a given desired criterion r depending on the system specification.

To design the robust controller for the system subject to a structured uncertainty, a new measurement (μ-synthesis) has been established by Doyle et al. [6] in 1982. Let $\Delta(jw) \in \mathbb{C}^{n\times n}$ define the structured uncertainty of the system with

$$\Delta = \{\mathrm{diag}[\delta_1 I_{r1},...,\delta_q I_{rq},\Delta_{n_1},...,\Delta_{n_k}] : \delta_i \in \mathbb{C}\ ,\ \Delta_{n_t} \in \mathbb{C}^{n_t\times n_t}\}$$

where

$$\sum_{i=1}^{q} r_i + \sum_{i=1}^{k} n_i = n$$

For fixed $w \in \mathbb{R}$, the structured singular value is formally defined as follows:

$$\begin{aligned}
\|M(jw)\|_{\mu} &\triangleq \mu_{\Delta}(M(jw)) \\
&\triangleq \frac{1}{\inf\{\bar{\sigma}(\Delta(jw)) \ , \ \Delta(jw) \in \Delta, \ \det(1 - M(jw)\Delta(jw)) = 0\}}
\end{aligned}$$

If $\det(1 - M(jw)\Delta(jw)) \neq 0$ for all $\Delta(jw) \in \Delta$, we define $\|M(jw)\|_{\mu} \triangleq 0$.

When $q = 1, \ k = 0, \ r_1 = n, \ \Delta = \{\delta I_n | \delta \in \mathbb{C}\}$, then

$$\mu_{\Delta}(M(jw)) = \rho(M(jw)) \triangleq \|M(jw)\|_s;$$

When $q = 0, \ k = 1, \ n_1 = n, \ \Delta = \mathbb{C}^{n \times n}$, then

$$\mu_{\Delta}(M(jw)) = \bar{\sigma}(M(jw)) \triangleq \|M(jw)\|_{\infty}.$$

Based on the definition of μ and the two special cases discussed above, we have

$$\|M(jw)\|_s \leq \|M(jw)\|_{\mu} \leq \|M(jw)\|_{\infty}$$

This fact is illustrated by the following example.

Example 1 Let

$$\Delta = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \ M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $\|M\|_s = 0, \ \|M\|_{\mu} = 0, \ \|M\|_{\infty} = 1.$

The sufficient conditions for robust stability and robust performance can be expressed in terms of structured singular value:

$$\|M\|_\mu < 1, \quad \|T(M, \Delta)\|_\mu < r$$

where $\|M\|_\mu = \sup_w \|M(jw)\|_\mu$.

The classical Nevanlinna-Pick interpolation problem is described below:

Let $\alpha_1, ..., \alpha_n \in \mathbb{D}$, $W_i \in \mathbb{C}^{n\times n}$. *Find all the analytic function* Φ *from* $\mathbb{D}$ *to* $\mathbb{C}^{n\times n}$ *satisfies*

$$\Phi(\alpha_i) = W_i, \ \forall i = 1, ..., n$$

and

$$\|\Phi\|_\infty \leq 1, \ \forall \alpha \in \mathbb{D}$$

We write down the Spectral-Nevanlinna-Pick interpolation problem as following:

Let $\alpha_1, ..., \alpha_n \in \mathbb{D}$, $W_i \in \mathbb{C}^{n\times n}$. *Find all the analytic function* F *from* $\mathbb{D}$ *to* $\mathbb{C}^{n\times n}$ *satisfies*

$$F(\alpha_i) = W_i, \ \forall i = 1, ..., n$$

and

$$\|F\|_\mu \leq 1, \ \forall \alpha \in \mathbb{D}$$

In this thesis we consider the situation of lower bound which is the spectral Nevanlinna-Pick problem. Only $n = 2$ case is considered and let the spectral radius with

$$\|A\|_s = \max\{|\lambda| : \lambda \text{ are the eigenvalues of A}\}$$

Define the spectral unit disc with

$$\Sigma = \{A \in \mathbb{C}^{2\times 2} : \|A\|_s \leq 1\}$$

which is 4-dimensional space in $\mathbb{C}$ and obviously unbounded, and non-convex, non-smooth set. For example:

Example 2 Let

$$A_1 = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & \frac{1}{2} \end{bmatrix}, \; A_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & \frac{1}{2} \end{bmatrix}$$

$$A_3 \triangleq \frac{1}{2}A_1 + (1 - \frac{1}{2})A_2 = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$\|A_3\|_s = \frac{3}{2} > 1$ thus $A_3 \notin \Sigma$, and Σ is not convex.

To study the interpolation problem in Σ is quite difficult. Suppose $A \in \mathbb{C}^{2\times 2}$ is similar to a companian matrix A_c :

$$A_c = \begin{bmatrix} 0 & 1 \\ -p & s \end{bmatrix}$$

with characteristic polynomial:

$$\det(\lambda I - A_c) = \lambda^2 - s\lambda + p$$

Define a set as following:

Definition 1.1 Define the symmetrized bidisc with

$$\Gamma = \{(s,p) : \lambda^2 - s\lambda + p = 0,\ |\lambda| \leq 1,\ \lambda \in \mathbb{C}\}$$

Where s and p are the sum and product of the roots of following equation

$$\lambda^2 - s\lambda + p = 0,\ |\lambda| \leq 1,\ \lambda \in \mathbb{C}$$

We can take (s,p) to be the coordinate of Γ. In other words, we can rewrite the symmetrized bidisc with

$$\Gamma = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_i| \leq 1,\ \lambda_i \in \mathbb{C},\ i = 1,2\}$$

In fact s and p are the trace and determinant of A_c respectively. Thus Γ is in the form:

$$\Gamma = \{(tr(A_c), det(A_c)) : A_c \text{ is the companion matrix of A},\ \text{A} \in \Sigma\}$$

Here Γ is in $\mathbb{C}^2$ which is a 2-dimensional space in $\mathbb{C}$. Γ is non-convex but bounded and compact set. Clearly some of the characteristics of Γ are better than Σ. The following two theorems show that we can study interpolation in Γ instead of Σ which was established by Agler and Young 1999.

Theorem 1.1 [2] *Let $\alpha_1, \ \alpha_2 \in \mathbb{D}$ be distinct and let $W_1, \ W_2 \in \Sigma$. Suppose that either both or neither of $W_1, \ W_2$ are scalar matrices. The following statements are equivalent.*

(1) There exists an analytic function $F : \mathbb{D} \to \Sigma$ such that $F(\alpha_i) = W_i$,

$i = 1, \ 2$.

(2) There exists an analytic function $f : \mathbb{D} \to \Gamma$ such that $f(\alpha_i) = (trW_i, detW_i)$,

$i = 1, \ 2$.

Theorem 1.2 [2] *Let $\alpha_1, \ \alpha_2 \in \mathbb{D}$ be distinct and let $W_1, \ W_2 \in \Sigma$ and suppose that $W_1 = cI$ but W_2 is not scalar. The following statements are equivalent.*

(1) There exists an analytic function $F : \mathbb{D} \to \Sigma$ such that $F(\alpha_i) = W_i$,

$i = 1, \ 2$.

(2) There exists an analytic function $f : \mathbb{D} \to \Gamma$ such that $f(\alpha_i) = (trW_i, detW_i)$,

$i = 1, \ 2$ and $f_2'(\alpha_1) = cf_1'(\alpha_1)$.

We write down the general spectral Nevanlinna-Pick interpolation problem as:

Given $\alpha_1, \ \alpha_2 \in \mathbb{D}$, and $(s_1, p_1), \ (s_2, p_2) \in \Gamma$, find an analytic function $\varphi : \mathbb{D} \to \Gamma$ with $\varphi(\lambda) = (s(\lambda), p(\lambda))$ satisfies

$$\varphi(\alpha_1) = (s_1, p_1), \ \varphi(\alpha_2) = (s_2, p_2).$$

In this thesis, we consider the following special flat problem:

Problem 1.1 *Given* $\alpha_2 \in \mathbb{D},\ \alpha_2 \neq 0$ *and* $(s_2, p_2) \in \Gamma$, *find an analytic function* $\varphi : \mathbb{D} \to \Gamma$ *with* $\varphi(\lambda) = (s(\lambda), p(\lambda))$ *satisfies*

$$\varphi(0) = (0, 0),\ \varphi(\alpha_2) = (s_2, p_2).$$

2 Mathematical Preliminaries

In this chapter we summarize some mathematical definitions and results which will be used in these thesis.

Denote $\mathbb{C}$ to be the complex space, and $\mathbb{D}$, $\overline{\mathbb{D}}$, $\mathbb{T}$ to be the unit disc, closed unit disc and unit circle, respectively. Consider the mapping $F: \mathbb{D} \to \mathbb{D}$ with

$$F_w(z) = \frac{z-w}{1-\bar{w}z}, \text{ where w} \in \mathbb{D}$$

which is called Möbius transformation.

Definition 2.1 For $w,\ z \in \mathbb{D}$, define the *Poincaré distance* with

$$d(z,w) = |\frac{z-w}{1-\bar{w}z}|$$

Definition 2.2 f is an *analytic function* in a domain Ω if it has a derivative at every point in Ω.

Theorem 2.1 [5] (Schwarz-Pick Lemma) *Suppose* $f: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *is an analytic function, then*

(1) $|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}| \leq |\frac{z-w}{1-z\bar{w}}| \quad \forall z,\ w \in \overline{\mathbb{D}}$

(2) $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \quad \forall z \in \overline{\mathbb{D}}$

If the equality holds then f *must be a conformal mapping.*

If f is an analytic function from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ then the Schwarz-Pick lemma shows that

$$d(f(z), f(w)) \leq d(z, w), \ \forall z, \ w \in \overline{\mathbb{D}}$$

Definition 2.3 Let Ω be an arbitrary set in $\mathbb{C}^n$ and taking all analytic function Q from $\mathbb{D}$ to Ω. Define the *Kobayashi distance* between $Q(z)$ and $Q(w)$ in Ω with

$$K_\Omega(Q(z), Q(w)) = \inf_Q \ d(z, w), \forall z, w \in \mathbb{D}$$

if the equility holds, Q is called Kobayashi extremal function(K-extremal function).

Definition 2.4 Let Ω be a domain in $\mathbb{C}^n$ and taking all analytic function F from Ω to $\mathbb{D}$. Define the *Carathéodory distance* between u and v in Ω with

$$C_\Omega(u, v) = \sup_F \ d(F(u), F(v)), \ \forall u, \ v \in \Omega$$

if the equility holds, F is called Carathéodory extremal function(C-extremal function).

By the Schwarz-Pick lemma, for any domain Ω *in* $\mathbb{C}^n$ we have [7]

$$C_\Omega \leq K_\Omega$$

By Lempert theorem [5] if Ω is convex then the equality hold.

Theorem 2.2 [8] (Maximum Modulus Theorem) *If f is defined and continuous on a closed-bounded set S and analytic on the interior of S, then the maximum of $|f(s)|$ on S is attained on the boundary of S, i.e.*

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|$$

where ∂S denotes the boundary of S.

We use $\mathrm{diag}[\lambda_1, ..., \lambda_n]$ to denote the diagonal matrix with entries $\lambda_1, ..., \lambda_n$.

Theorem 2.3 [4] *Let h be a analytic bounded-valued function on $\mathbb{D}^2$ and $|h(\lambda_1, \lambda_2)| \leq 1$, for all $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ if and only if there exists an unitary matrix*

$$\mathcal{R}_h = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

such that

$$h(\lambda_1, \lambda_2) = C \ \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2](I_3 - A \ \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2])^{-1} B + D$$

for all $(\lambda_1, \lambda_2) \in \mathbb{D}^2$.

3 Main Results

In this chapter we discuss the properties of Γ, the ideal to construct φ, and given a explicit 2D realization.

3.1 Properties of Γ

In this section we are interesting the geometric properties about Γ. First we are concerned what kinds of s and p in Γ.

Theorem 3.1 *The following statements are equivalent:*

(1) $(s,p) \in \Gamma$

(2) $|s-\bar{s}p| \leq 1-|p|^2$ *and* $|s| \leq 2$

(3) $2|s-\bar{s}p| + |s^2-4p| \leq 4-|s|^2 \quad , \quad |p| \leq 1$

Proof. Write $s = \lambda_1 + \lambda_2$, $p = \lambda_1\lambda_2$ and taking the polar form with

$\lambda_1 = r_1 e^{i\theta_1}$, $\lambda_2 = r_2 e^{i\theta_2}$, $s = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$, $p = r_1 r_2 e^{i(\theta_1+\theta_2)}$

, $0 \leq r_1 \leq 1$, $0 \leq r_2 \leq 1$.

For (1) $\Rightarrow$ (2).

Consider

$$\begin{aligned} |s-\bar{s}p| - (1-|p|^2) &= |r_1 e^{i\theta_1} + r_2 e^{i\theta_2} - (r_1 e^{-i\theta_1} + r_2 e^{-i\theta_2}) r_1 r_2 e^{i(\theta_1+\theta_2)}| - (1 - r_1^2 r_2^2) \\ &= |r_1 e^{i\theta_1}(1-r_2^2) + r_2 e^{i\theta_2}(1-r_1^2)| - (1-r_1^2 r_2^2) \\ &\leq |r_1 e^{i\theta_1}(1-r_2^2)| + |r_2 e^{i\theta_2}(1-r_1^2)| - (1-r_1^2 r_2^2) \end{aligned}$$

$$
\begin{aligned}
&= r_1(1-r_2^2)+r_2(1-r_1^2)-(1-r_1^2r_2^2) \\
&= (1-r_1r_2)[r_1+r_2-(1+r_1r_2)] \\
&\leq 0
\end{aligned}
$$

Since

$$(1+r_1r_2)^2-(r_1+r_2)^2=(1-r_1^2)(1-r_2^2)\geq 0$$

which implies that $r_1+r_2-(1+r_1r_2)\leq 0$.

For (2) $\Rightarrow$ (1). By $0\leq |s-\bar{s}p|\leq 1-|p|^2=(1-r_1^2r_2^2)$ we have $r_1r_2\leq 1$.

Suppose $r_1\geq 1$ then $r_2\leq \frac{1}{r_1}<1$

$$
\begin{aligned}
0 &\geq |s-\bar{s}p|-(1-|p|^2) \\
&= |r_1e^{i\theta_1}+r_2e^{i\theta_2}-(r_1e^{-i\theta_1}+r_2e^{-i\theta_2})r_1r_2e^{i(\theta_1+\theta_2)}|-(1-r_1^2r_2^2) \\
&= |r_1e^{i\theta_1}(1-r_2^2)+r_2e^{i\theta_2}(1-r_1^2)|-(1-r_1^2r_2^2) \\
&\geq |r_1e^{i\theta_1}(1-r_2^2)|-|(-1)r_2e^{i\theta_2}(1-r_1^2)|-(1-r_1^2r_2^2) \\
&= |r_1e^{i\theta_1}(1-r_2^2)|+|r_2e^{i(\theta_2+\pi)}(1-r_1^2)|-(1-r_1^2r_2^2) \\
&= |r_1(1-r_2^2)|-|r_2(1-r_1^2)|-(1-r_1^2r_2^2) \\
&= r_1(1-r_2^2)-r_2(1-r_1^2)-(1-r_1^2r_2^2) \\
&= (1-r_1r_2)(r_1+r_2-1-r_2r_2) \\
&= (1-r_1r_2)(1-r_2)(r_1-1)
\end{aligned}
$$

But $1-r_1r_2 \geq 0$, $r_1 - 1 > 0$, $1-r_2 > 0$ which is contradiction to $0 \geq |s-\bar{s}p| - (1-|p|^2)$. Hence $r_1 \leq 1$. Similarly, $r_2 \leq 1$. Thus $(s,p) \in \Gamma$.

For (1) $\Rightarrow$ (3). Consider

$$\begin{aligned}
2|s-\bar{s}p| + |s^2-4p| - (4-|s|^2) &= 2|r_1e^{i\theta_1} + r_2e^{i\theta_2} - (r_1e^{-i\theta_1} + r_2e^{-i\theta_2})(r_1r_2e^{i(\theta_1+\theta_2)})| \\
&\quad +|(r_1e^{i\theta_1} + r_2e^{i\theta_2})^2 - 4r_1r_2e^{i(\theta_1+\theta_2)}| \\
&\quad +|r_1e^{i\theta_1} + r_2e^{i\theta_2}|^2 - 4 \\
&\leq 2|r_1e^{i\theta_1}(1-r_2^2)| + 2|r_2e^{i\theta_2}(1-r_1^2)| + |r_1e^{i\theta_1} - r_2e^{i\theta_2}|^2 \\
&\quad +|r_1e^{i\theta_1} + r_2e^{i\theta_2}|^2 - 4 \\
&= 2r_1(1-r_2^2) + 2r_2(1-r_1^2) + 2r_1^2 + 2r_2^2 - 4 \\
&= -2(1-r_1)(1-r_2)(r_1+r_2+2) \\
&\leq 0
\end{aligned}$$

For (3) $\Rightarrow$ (1). Suppose $r_1 > 1$ and by $|p| \leq 1$ we have $r_2 \leq \frac{1}{r_1} < 1$.

$$\begin{aligned}
2|s-\bar{s}p| + |s^2-4p| - (4-|s|^2) &= 2|r_1e^{i\theta_1} + r_2e^{i\theta_2} - (r_1e^{-i\theta_1} + r_2e^{-i\theta_2})(r_1r_2e^{i(\theta_1+\theta_2)})| \\
&\quad +|(r_1e^{i\theta_1} + r_2e^{i\theta_2})^2 - 4r_1r_2e^{i(\theta_1+\theta_2)}| \\
&\quad +|r_1e^{i\theta_1} + r_2e^{i\theta_2}|^2 - 4
\end{aligned}$$

$$\begin{aligned}
&= 2|r_1e^{i\theta_1}(1-r_2^2)+r_2e^{i\theta_2}(1-r_1^2)|+|r_1e^{i\theta_1}-r_2e^{i\theta_2}|^2 \\
&\quad +|r_1e^{i\theta_1}+r_2e^{i\theta_2}|^2-4 \\
&\geq 2|r_1(1-r_2^2)|-2|r_2(1-r_1^2)|+2r_1^2+2r_2^2-4 \\
&= 2r_1(1-r_2^2)-2r_2(r_1^2-1)+2r_1^2+2r_2^2-4 \\
&= -2(r_1-1)(r_2-1)(r_1+r_2+2) \\
&\geq 0
\end{aligned}$$

which contradiction to $2|s-\bar{s}p|+|s^2-4p|-(4-|s|^2)\leq 0$. Thus $r_1\leq 1$. Similarly $r_2\leq 1$. Hence $(s,p)\in\Gamma$ and we complete the proof. ■

Define the interior symmetrized bidisc with

$$int(\Gamma)=\{(\lambda_1+\lambda_2,\lambda_1\lambda_2):\ \lambda_i\in\mathbb{C},\ |\lambda_i|<1\}$$

then $(s,p)\in int(\Gamma)$ if and only if $|s-\bar{s}p|<1-|p|^2$.

The following lemma shows that for all line-segments connected $(0,0)$ and any other points in Γ are still lies in Γ. Also we can say that there are no holes in Γ.

Lemma 3.1 *If* $(s,p)\in\Gamma$, *then for any* $0<t<1$, $(ts,tp)\in\Gamma$

Proof. Since $(s,p)\in\Gamma$. Write $s=\lambda_1+\lambda_2$, $p=\lambda_1\lambda_2$ and taking the polar form with $\lambda_1=r_1e^{i\theta_1}$, $\lambda_2=r_2e^{i\theta_2}$, $0\leq r_1\leq 1$, $0\leq r_2\leq 1$

and $s = \lambda_1 + \lambda_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2},\ p = \lambda_1 \lambda_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

Claim (ts, tp) satisfy $|ts - t^2 \overline{s} p| \leq 1 - t^2 |p|^2$. Consider

$$\begin{aligned}
\left|ts - t^2 \overline{s} p\right| &= t\left|r_1 e^{i\theta_1} + r_2 e^{i\theta_2} - t(r_1 e^{-i\theta_1} + r_2 e^{-i\theta_2}) r_1 r_2 e^{i(\theta_1+\theta_2)}\right| \\
&= t\left|r_1 e^{i\theta_1} + r_2 e^{i\theta_2} - t r_1^2 r_2 e^{i\theta_2} - t r_1 r_2^2 e^{i\theta_1}\right| \\
&= t\left|r_1 e^{i\theta_1}(1 - t r_2^2) + r_2 e^{i\theta_2}(1 - t r_1^2)\right| \\
&\leq t\left|r_1 e^{i\theta_1}(1 - t r_2^2)\right| + t\left|r_2 e^{i\theta_2}(1 - t r_1^2)\right| \\
&= t r_1 (1 - t r_2^2) + t r_2 (1 - t r_1^2) \\
&= t r_1 - t^2 r_1 r_2^2 + t r_2 - t^2 r_1^2 r_2 \\
&= t(r_1 + r_2)(1 - t r_1 r_2)
\end{aligned}$$

Hence

$$\begin{aligned}
\left|ts - t^2 \overline{s} p\right| - (1 - t^2 |p|^2) &\leq t(r_1 + r_2)(1 - t r_1 r_2) - (1 - t^2 |p|^2) \\
&= t(r_1 + r_2)(1 - t r_1 r_2) - (1 - t^2 r_1^2 r_2^2) \\
&= (1 - t r_1 r_2)(t r_1 + t r_2 - (1 + t r_1 r_2))
\end{aligned}$$

Now $(1 - t r_1 r_2) \geq 0$, we claim $t r_1 + t r_2 - (1 + t r_1 r_2) \leq 0$.

Consider

$$\begin{aligned}(t(r_1+r_2))^2-(1+tr_1r_2)^2 &= t^2(r_1^2+2r_1r_2+r_2^2)-1-2tr_1r_2-t^2r_1^2r_2^2 \\ &= t^2r_1^2+t^2r_2^2-1-t^2r_1^2r_2^2+2tr_1r_2(t-1) \\ &< tr_1^2+tr_2^2-1-t^2r_1^2r_2^2+2tr_1r_2(t-1) \\ &= -(1-tr_1^2)(1-tr_2^2)+2tr_1r_2(t-1) \\ &< 0\end{aligned}$$

Thus $t(r_1+r_2)-(1+tr_1r_2)<0$.

And $|ts-t^2\overline{s}p|-(1-t^2|p|^2)\leq(1-tr_1r_2)(tr_1+tr_2-(1+tr_1r_2))<0$.

Hence $(ts,tp)\in\Gamma$. ■

3.2 Ideal to Construct φ

Define the function G_w from $int(\Gamma)$ to $\mathbb{D}$ with[1]:

$$G_w(s,p) = \frac{2p - ws}{2 - \bar{w}s}, \ w \in \overline{\mathbb{D}}$$

is an analytic function. And we have the following result:

Theorem 3.2 *Let* $(s_2, p_2) \in int(\Gamma)$ *then*

$$C_{int(\Gamma)}((0,0),(s_2,p_2)) = \sup_{|w|=1} |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}|$$

Proof. By the definition and maximum modulus theorem, we have

$$C_{int(\Gamma)}((0,0),(s_2,p_2)) = \sup_{|w|\le 1} |\frac{\frac{2p_2 - ws_2}{2-\bar{w}s_2} - 0}{1 - \frac{2p_2 - ws_2}{2-\bar{w}s_2}0}| = \sup_{|w|\le 1} |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}| = \sup_{|w|=1} |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}|$$

■

From Theorem 3.2, if the equality holds at $|w_0| = 1$, i.e.

$$C_{int(\Gamma)}((0,0),(s_2,p_2)) = d(G_{w_0}(0,0), G_{w_0}(s_2,p_2)) = |\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}|$$

then G_{w_0} is a C-extremal function.

Definition 3.1 *If* $C_{int(\Gamma)} = K_{int(\Gamma)}$*, Define* φ *is a K-extremal function from* $\mathbb{D}$ *to* Γ.

Now we have the problem: How to construct φ ?

Let

$$\beta_2 \triangleq G_w(s_2, p_2) = \frac{2p_2 - ws_2}{2 - \bar{w}s_2}.$$

Clearly $|\beta_2| \leq 1$. By the K-extremal property of φ we have

$$G_w \circ \varphi(\alpha_2) = \beta_2.$$

Since φ and G_w are analytic functions, so we have $G_w \circ \varphi$ is also an analytic function from $\mathbb{D}$ to $\mathbb{D}$.

From the Nevanlinna-Pick theorem we have the sufficient and necessary condition for the existence of an analytic function $G_w \circ \varphi$ satisfies $G_w \circ \varphi(0) = 0$, $G_w \circ \varphi(\alpha_2) = \beta_2$ which required the Pick matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & \frac{1-|\beta_2|^2}{1-|\alpha_2|^2} \end{bmatrix} \geq 0$$

is positive semi-define.

Now, we are concerned the sufficient and necessary conditions of φ. The following necessary condition was established by Agler and Young 1999.

Theorem 3.3 [3] *Let $\alpha_2 \in \mathbb{D}$ and $(s_2, p_2) \in int(\Gamma)$, if there exist an analytic function $\varphi : \mathbb{D} \to int(\Gamma)$ satisfies $\varphi(0) = (0, 0)$, $\varphi(\alpha_2) = (s_2, p_2)$.*

Then

$$\begin{bmatrix} 2 & 2-ws_2 \\ 2-\bar{w}\bar{s}_2 & \frac{2(1-\bar{p}_2p_2)-\bar{w}(\bar{s}_2-\bar{p}_2s_2)-w(s_2-\bar{s}_2p_2)}{1-\bar{\alpha}_2\alpha_2} \end{bmatrix} \geq 0$$

To construct the sufficient condition of φ we consider the following theorem:

Theorem 3.4 *Let* $C_{int(\Gamma)}((0,0),\ (s_2,p_2)) = K_{int(\Gamma)}((0,0),\ (s_2,p_2))$ *where C-extremal function and K-extremal function are* $G_{w_0} : int(\Gamma) \to \mathbb{D}$ *and* $\varphi : \mathbb{D} \to \Gamma$*, then* $G_{w_0} \circ \varphi : \mathbb{D} \to \mathbb{D}$ *is analytic and*

(1) $G_{w_0} \circ \varphi = id_{\mathbb{D}}$ *(up to Möbius transformation).*

(2) φ *is isometric transformation from* $(\mathbb{D}, d)$ *to* $(\varphi(\mathbb{D}), C_{int(\Gamma)})$.

where $(\varphi(\mathbb{D}), C_{int(\Gamma)})$ *is called totally geodesic disc of* $int(\Gamma)$.

Proof. Since G_{w_0} is a C-extremal function, φ is a K-extremal function. Clearly $G_{w_0} \circ \varphi$ is an analytic function from $\mathbb{D}$ to $\mathbb{D}$. Suppose $\varphi(0) = (0,0),\ \varphi(\alpha_2) = (s_2, p_2)$ then

$|\alpha_2| = d(0,\alpha_2) = K_{int(\Gamma)}(\varphi(0),\varphi(\alpha_2)) = K_{int(\Gamma)}((0,0),\ (s_2,p_2)) = C_{int(\Gamma)}((0,0),\ (s_2,p_2)) = d(G_{w_0}(0,0), G_{w_0}(s_2,p_2)) = d(0,\beta_2) = |\beta_2|$

Hence $G_{w_0} \circ \varphi$ is an isometric transformation from $(\mathbb{D}, d)$ *to* $(\varphi(\mathbb{D}), C_{int(\Gamma)})$.

Also $G_{w_0} \circ \varphi = id_{\mathbb{D}}$. ■

We consider the situation of $G_{w_0} \circ \varphi(\lambda) = \lambda$. Let M be a Möbius transformation such that $M(\beta_2) = \alpha_2$ then

$$M \circ G_{w_0} \circ \varphi(\alpha_2) = M(\beta_2) = \alpha_2$$

then $M \circ G_{w_0} \circ \varphi$ is an identity function, without lose any generality. Let $M = I$ then $\alpha_2 = \beta_2$. In other words, let $\varphi(\lambda) = (s(\lambda), p(\lambda))$ so we have

$$G_{w_0}(s(\lambda), p(\lambda)) = \frac{2p(\lambda) - w_0 s(\lambda)}{2 - \bar{w}_0 s(\lambda)} = \lambda$$

and

$$s(\lambda) = 2\frac{p(\lambda) - \lambda}{w_0 - \bar{w}_0 \lambda}$$

where $s(\cdot)$ must be analytic, so $p(\cdot)$ have to satisfies $p(w_0^2) = w_0^2$, $p(\cdot)$ is analytic, $p(\alpha_2) = p_2$, $p(0) = 0$ and $|p(\lambda)| \leq 1,\ \forall \lambda \in \mathbb{D}$.

If $K_{int(\Gamma)} = C_{int(\Gamma)}$ by

$$\beta_1 = 0,\ \beta_2 = \frac{2p_2 - ws_2}{2 - \bar{w}s_2}, d(0, \beta_2) = |\beta_2| = |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}|$$

and by theorem 3.2

$$C_{int(\Gamma)}((0,0),(s_2,p_2)) = \sup_{|w|=1} |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}|.$$

If there exists $w_0 \in \mathbb{T}$ such that

$$|\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}| = \sup_{|w|=1} |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}|$$

Lemma 3.2 *Let $w_0 \in \mathbb{T}$ and satisfies*

$$|\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}| = \sup_{|w|=1} |\frac{2p_2 - ws_2}{2 - \bar{w}s_2}|$$

then

$$|\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}| = \frac{|4p_2 - s_2^2| + 2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}$$

Proof. Claim

$$|\frac{2p_2 - w s_2}{2 - w\bar{s}_2} - \frac{4p_2 - s_2^2}{4 - |s_2|^2}| \leq \frac{2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}$$

and

$$\begin{aligned}
\frac{2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2} - |\frac{2p_2 - w s_2}{2 - w\bar{s}_2} - \frac{4p_2 - s_2^2}{4 - |s_2|^2}| &= \frac{2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2} - 2|\frac{(2w - s_2)(p_2\bar{s}_2 - s_2)}{(2 - w\bar{s}_2)(4 - |s_2|^2)}| \\
&= o, \forall w \in \mathbb{T}
\end{aligned}$$

We have

$$|\frac{2p_2 - w s_2}{2 - w\bar{s}_2}| - |\frac{4p_2 - s_2^2}{4 - |s_2|^2}| \leq |\frac{2p_2 - w s_2}{2 - w\bar{s}_2} - \frac{4p_2 - s_2^2}{4 - |s_2|^2}| = \frac{2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}$$

Hence

$$|\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}| = \sup_{|w|=1} |\frac{2p_2 - w s_2}{2 - \bar{w} s_2}| = \frac{|4p_2 - s_2^2| + 2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}.$$

■

From the lemma 3.2 and theorem 3.4 we have the following identity:

$$|\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}| = |\beta_2| = \frac{|4p_2 - s_2^2| + 2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}$$

3.3 Realization of Symmetrized Bidisc Γ

In this section we characterize an analytic function φ and give a realization of φ.

Theorem 3.5 *Let* $\alpha_2 \in \mathbb{D}$, $(s_2, p_2) \in int(\Gamma)$, $\alpha_2 \neq 0$, *and if there exists* $w_0 \in \mathbb{T}$ *satisfies the following*

$$|\frac{2p_2 - w_0 s_2}{2 - \bar{w}_0 s_2}| = |\alpha_2| = \frac{|4p_2 - s_2^2| + 2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}.$$

Then there exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ *such that*

$$\varphi(0) = (0, 0)$$

$$\varphi(\alpha_2) = (s_2, p_2)$$

where $\varphi(\lambda) = (s(\lambda), p(\lambda))$.

$$s(\lambda) = \frac{2\alpha\lambda}{1 - \bar{\beta}\lambda}, p(\lambda) = \frac{\gamma\lambda(\lambda - \beta)}{1 - \bar{\beta}\lambda}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, *and* $\alpha = \frac{\bar{w}_0 k}{\bar{a}}$, $k = w_0^2 \bar{b} - a \in \mathbb{R}$, $\beta = -\frac{b}{a}$, $\gamma = \bar{w}_0^2 \frac{a}{\bar{a}}$ *with* $a = 1 - w_0^2 p_0 \bar{\alpha}_2$, $b = w_0^2 p_0 - \alpha_2$, $p_0 = \frac{p_2}{\alpha_2}$, $a + b\bar{w}_0^2 \in \mathbb{R}$, $|k| = |a| - |b|$.

Proof. First we consider the following identities:

(i) $|\gamma| = |\bar{w}_0^2 \frac{a}{\bar{a}}| = 1.$

(ii) $\gamma\bar{\alpha} = \bar{w}_0^2 \frac{a}{\bar{a}} \overline{\frac{\bar{w}_0 k}{\bar{a}}} = \frac{\bar{w}_0 k}{\bar{a}} = \alpha.$

(iii) $|\alpha|+|\beta|=|\frac{\bar{w}_0 k}{\bar{a}}|+|-\frac{b}{a}|=\frac{1}{|a|}(|k|+|b|)=1.$

To check the interpolation.

$$\begin{aligned}
p(\lambda) &= \frac{\gamma\lambda(\lambda-\beta)}{1-\bar{\beta}\lambda} \\
&= \bar{w}_0^2\frac{a}{\bar{a}}\frac{\lambda(\lambda+\frac{b}{a})}{1+\frac{\bar{b}}{\bar{a}}\lambda} \\
&= \bar{w}_0^2\lambda\frac{a\lambda+b}{\bar{b}\lambda+\bar{a}} \\
&= \bar{w}_0^2\lambda\frac{(1-w_0^2p_0\bar{\alpha}_2)\lambda+(w_0^2p_0-\alpha_2)}{(\bar{w}_0^2\bar{p}_0-\bar{\alpha}_2)\lambda+(1-\bar{w}_0^2\bar{p}_0\alpha_2)} \\
&= \lambda\frac{(\lambda-\alpha_2)\bar{w}_0^2+p_0(1-\bar{\alpha}_2\lambda)}{1-\bar{\alpha}_2\lambda+\bar{p}_0\bar{w}_0^2(\lambda-\alpha_2)}
\end{aligned}$$

Thus

$$p(\alpha_2)=\alpha_2 p_0=p_2$$

and

$$s(\alpha_2)=2\frac{p(\alpha_2)-\alpha_2}{w_0-\bar{w}_0\alpha_2}=2\frac{p_2-\beta_2}{w_0-\bar{w}_0\beta_2}=2\frac{p_2-\frac{2p_2-w_0s_2}{2-\bar{w}_0s_2}}{w_0-\bar{w}_0\frac{2p_2-w_0s_2}{2-\bar{w}_0s_2}}=s_2\frac{w_0-p_2\bar{w}_0}{w_0-p_2\bar{w}_0}=s_2.$$

Since the pole of $p(\cdot)$ is $\frac{1}{\bar{\beta}}\notin\mathbb{D}$, and

$$p(w_0^2)=w_0^2\frac{1-\alpha_2\bar{w}_0^2+p_0-p_0\bar{\alpha}_2w_0^2}{1-\bar{\alpha}_2w_0^2+\bar{p}_0-\bar{p}_0\alpha_2\bar{w}_0^2}=w_0^2\frac{a+\bar{w}_0^2b}{\bar{a}+w_0^2\bar{b}}=w_0^2.$$

Thus φ is an analytic function.

To proof $\varphi(\lambda) \in \Gamma$. Let $u(\lambda) = 1 - \bar{\beta}\lambda$ and $k(\lambda) = \lambda - \beta$. Check $|s(\lambda)| \leq 2$

$$\begin{aligned}
4 - |s(\lambda)|^2 &= 4 - |\frac{2\alpha\lambda}{1-\bar{\beta}\lambda}|^2 \\
&= \frac{4}{|u(\lambda)|^2}(1 - \beta\bar{\lambda} - \bar{\beta}\lambda + |\beta|^2|\lambda|^2 - |\alpha|^2|\lambda|^2) \\
&= \frac{4}{|u(\lambda)|^2}(1 - \beta\bar{\lambda} - \bar{\beta}\lambda + |\beta||\lambda|^2 - |\alpha||\lambda|^2) \\
&= \frac{4}{|u(\lambda)|^2}(|\beta| - \beta\bar{\lambda} - \bar{\beta}\lambda + |\beta||\lambda|^2 + |\alpha| - |\alpha||\lambda|^2) \\
&\geq \frac{4}{|u(\lambda)|^2}(|\beta| - 2|\beta||\lambda| + |\beta||\lambda|^2 + |\alpha| - |\alpha||\lambda|^2) \\
&= \frac{4}{|u(\lambda)|^2}(|\beta|(1-|\lambda|)^2 + |\alpha|(1-|\lambda|^2)) \\
&\geq 0
\end{aligned}$$

Hence $|s(\lambda)| \leq 2$. Check $\varphi(\lambda) \in \Gamma$.

$$\begin{aligned}
1 - |p(\lambda)|^2 - |s(\lambda) - \overline{s(\lambda)}p(\lambda)| &= 1 - \frac{\gamma\lambda k(\lambda)}{u(\lambda)}\frac{\overline{\gamma\lambda k(\lambda)}}{\overline{u(\lambda)}} - |\frac{2\alpha\lambda}{1-\bar{\beta}\lambda} - \frac{\overline{2\alpha\lambda}}{\overline{u(\lambda)}}\frac{\gamma\lambda k(\lambda)}{u(\lambda)}| \\
&= \frac{1}{|u(\lambda)|^2}\{|u(\lambda)|^2 - |\gamma\lambda k(\lambda)|^2 - 2|\alpha||\overline{u(\lambda)}\lambda - k(\lambda)|\lambda|^2|\} \\
&= \frac{1}{|u(\lambda)|^2}\{(1-|\lambda|^2)(1 - \beta\bar{\lambda} - \bar{\beta}\lambda + |\lambda|^2) - 2|\alpha||\lambda|(1-|\lambda|^2)\} \\
&= \frac{1}{|u(\lambda)|^2}(1-|\lambda|^2)(1 - \beta\bar{\lambda} - \bar{\beta}\lambda + |\lambda|^2 - 2|\alpha||\lambda|) \\
&= \frac{1}{|u(\lambda)|^2}(1-|\lambda|^2)(1 - \beta\bar{\lambda} - \bar{\beta}\lambda + |\lambda|^2 - 2|\lambda| + 2|\beta||\lambda|) \\
&= \frac{1}{|u(\lambda)|^2}(1-|\lambda|^2)(1-|\lambda|)^2(2|\beta||\lambda| - \beta\bar{\lambda} - \bar{\beta}\lambda) \\
&\geq 0
\end{aligned}$$

Hence $\forall\lambda \in \mathbb{D}, \varphi(\lambda) \in \Gamma$. ∎

We given an explicit 2D realization in the following theorem.

Theorem 3.6 *If* $\varphi(\lambda) = (\frac{2\alpha\lambda}{1-\overline{\beta}\,\lambda}, \gamma\lambda\frac{\lambda-\beta}{1-\overline{\beta}\,\lambda}) = (s(\lambda), p(\lambda)) \in int(\Gamma)$, *where* $\alpha,\ \beta,\ \gamma \in \mathbb{C},\ \ |\alpha| + |\beta| = 1,\ \gamma\overline{\alpha} = \alpha,\ \ |\gamma| = 1$

Let

$$h(\lambda_1, \lambda_2) = \frac{2\lambda_2 p(\lambda_1) - s(\lambda_1)}{2 - \lambda_2 s(\lambda_1)}$$

then there exists a unitary matrix

$$\mathcal{R}_h = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ |\alpha\beta|^{\frac{1}{2}} & \overline{\beta} & \frac{-\alpha\beta}{|\alpha|^{\frac{1}{2}}|\beta|} \\ |\beta|^{\frac{1}{2}} & -|\alpha|^{\frac{1}{2}}\frac{\bar{\beta}}{|\beta|} & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} -\alpha & \frac{-\alpha\overline{\beta}}{|\alpha\beta|^{\frac{1}{2}}} & \frac{-\gamma\beta}{|\beta|^{\frac{1}{2}}} \end{bmatrix},\ D = 0$$

such that

$$h(\lambda_1, \lambda_2) = C\ \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2](I_3 - A\ \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2])^{-1}B + D$$

for all $(\lambda_1, \lambda_2) \in \mathbb{D}^2$.

Proof.

$$h(\lambda_1, \lambda_2) = \frac{2\lambda_2 p(\lambda_1) - s(\lambda_1)}{2 - \lambda_2 s(\lambda_1)} = \frac{\lambda_1[\gamma\lambda_2(\lambda_1 - \beta) - \alpha]}{1 - \overline{\beta}\lambda_1 - \alpha\lambda_1\lambda_2}$$

Let

$m \triangleq m(\lambda_1,\lambda_2) = \overline{\beta}\lambda_1 + \alpha\lambda_1\lambda_2,\ k \triangleq k(\lambda_1) = \lambda_1 - \beta,\ \ u \triangleq u(\lambda_1,\lambda_2) = 1 - m,$

$h \triangleq h(\lambda_1,\lambda_2)$

Consider

$1 - h(\lambda_1,\lambda_2)\overline{h(\lambda_1,\lambda_2)}$

$= (1-|\lambda_1|^2) + |\lambda_1|^2\,(1 - \frac{\gamma\lambda_2 k-\alpha}{u}\frac{\overline{\gamma\lambda_2 k}-\overline{\alpha}}{\overline{u}})$

$= (1-|\lambda_1|^2) + \frac{|\lambda_1|^2}{|u|^2}[u\overline{u} - (\gamma\lambda_2 k - \alpha)(\overline{\gamma\lambda_2 k} - \overline{\alpha})]$

$= (1-|\lambda_1|^2) + \frac{|\lambda_1|^2}{|u|^2}[1 - m - \overline{m} + m\overline{m} - |\lambda_2 k|^2 + \alpha k\lambda_2 + \overline{\alpha\lambda_2 k} - |\alpha|^2]\quad (Since\ \gamma\overline{\alpha} = \alpha)$

$= (1-|\lambda_1|^2) + \frac{|\lambda_1|^2}{|u|^2}[(1-|\lambda_2|^2)\,|k|^2 - |k|^2 + 1 - m - \overline{m} + m\overline{m} + \alpha k\lambda_2 + \overline{\alpha\lambda_2 k} - |\alpha|^2]$

$= (1-|\lambda_1|^2) + \frac{|\lambda_1|^2}{|u|^2}[(1-|\lambda_2|^2)\,|k|^2 - |\lambda_1|^2\,(1-|\beta|^2 - \overline{\alpha\beta\lambda_2} - \alpha\beta\lambda_2)$

$\quad + (1-|\beta|^2 - \overline{\alpha\beta\lambda_2} - \alpha\beta\lambda_2 - |\alpha|^2) + |\alpha\lambda_1\lambda_2|^2]$

$= (1-|\lambda_1|^2) + \frac{|\lambda_1|^2}{|u|^2}[(1-|\lambda_1|^2)(|\alpha\beta| - \overline{\alpha\beta\lambda_2} - \alpha\beta\lambda_2 + |\alpha\beta|\,|\lambda_2|^2)$

$\quad + (1-|\lambda_2|^2)(|\alpha\beta| - |\alpha\lambda_1|^2 - |\alpha\beta|\,|\lambda_1|^2 + |\lambda_1|^2 - \beta\overline{\lambda_1} - \overline{\beta}\lambda_1 + |\beta|^2)]$

$= (1-|\lambda_1|^2) + \frac{|\lambda_1|^2}{|u|^2}[(1-|\lambda_1|^2)\,|\alpha\beta|\,(1 - \frac{\alpha\beta}{|\alpha\beta|}\lambda_2)\overline{(1 - \frac{\alpha\beta}{|\alpha\beta|}\lambda_2)}$

$\quad + (1-|\lambda_2|^2)\,|\beta|\,(1 - \frac{\overline{\beta}}{|\beta|}\lambda_1)\overline{(1 - \frac{\overline{\beta}}{|\beta|}\lambda_1)}]$

Thus

$$\begin{aligned} 1 - h(\lambda_1,\lambda_2)\overline{h(\lambda_1,\lambda_2)} &= (1-|\lambda_1|^2)[1 + \frac{|\lambda_1|^2}{|u|^2}\,|\alpha\beta|\,(1 - \frac{\alpha\beta}{|\alpha\beta|}\lambda_2)\overline{(1 - \frac{\alpha\beta}{|\alpha\beta|}\lambda_2)}] \\ &\quad + \frac{|\lambda_1|^2}{|u|^2}(1-|\lambda_2|^2)\,|\beta|\,(1 - \frac{\overline{\beta}}{|\beta|}\lambda_1)\overline{(1 - \frac{\overline{\beta}}{|\beta|}\lambda_1)} \\ &\geq 0 \end{aligned}$$

Hence $|h| \leq 1$.

Let

$$g^1 \triangleq g^1(\lambda_1, \lambda_2) = \begin{bmatrix} 1 \\ \frac{\lambda_1}{u} |\alpha\beta|^{\frac{1}{2}} (1 - \frac{\alpha\beta}{|\alpha\beta|}\lambda_2) \end{bmatrix}, \; g^2 \triangleq g^2(\lambda_1, \lambda_2) = \left[\frac{\lambda_1}{u} |\beta|^{\frac{1}{2}} (1 - \frac{\overline{\beta}}{|\beta|}\lambda_1)\right]$$

Then

$$\begin{aligned} 1 - h(\lambda_1, \lambda_2)\overline{h(\lambda_1, \lambda_2)} &= (1 - |\lambda_1|^2) < g^1, g^1 > +(1 - |\lambda_2|^2) < \; g^2, \; g^2 > \\ &= \; < g^1, g^1 > - < \lambda_1 g^1, \lambda_1 g^1 > + < g^2, \; g^2 > - < \lambda_2 g^2, \lambda_2 g^2 > \end{aligned}$$

Hence

$$< \lambda_1 g^1, \lambda_1 g^1 > + < \lambda_2 g^2, \lambda_2 g^2 > +1 =< g^1, g^1 > + < g^2, \; g^2 > + < h, h >$$

We can find

$$A = \begin{bmatrix} 0 & 0 & 0 \\ |\alpha\beta|^{\frac{1}{2}} & \overline{\beta} & \frac{-\alpha\beta}{|\alpha|^{\frac{1}{2}}|\beta|} \\ |\beta|^{\frac{1}{2}} & -|\alpha|^{\frac{1}{2}} \frac{\bar{\beta}}{|\beta|} & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} -\alpha & \frac{-\alpha\overline{\beta}}{|\alpha\beta|^{\frac{1}{2}}} & \frac{-\gamma\beta}{|\beta|^{\frac{1}{2}}} \end{bmatrix}, \; D = 0$$

such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \lambda_1 g^1 \\ \lambda_2 g^2 \\ 1 \end{bmatrix} = \begin{bmatrix} g^1 \\ g^2 \\ h \end{bmatrix}$$

And

$$A \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} g^1 \\ g^2 \end{bmatrix} + B = \begin{bmatrix} g^1 \\ g^2 \end{bmatrix}$$

Thus

$$\begin{bmatrix} g^1 \\ g^2 \end{bmatrix} = \left(I_3 - A \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \right)^{-1} B$$

$$
\begin{aligned}
h &= C \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} g^1 \\ g^2 \end{bmatrix} + D \\
&= C \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \left(I_3 - A \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \right)^{-1} B \\
&= C \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2](I_3 - A \ \mathrm{diag}[\lambda_1, \lambda_1, \lambda_2])^{-1} B + D
\end{aligned}
$$

for all $(\lambda_1, \lambda_2) \in \mathbb{D}^2$. ■

4 Concluding Remarks

In this thesis we discuss the Schwarz lemma on symmetrized bidisc under the condition $C_{int(\Gamma)}((0,0),(s_2,p_2)) = K_{int(\Gamma)}((0,0),(s_2,p_2))$. We given an analytic function φ and find a realization in the special flat Nevanlinna-Pick problem.

In the future, there are some natural questions suggested by the present results.

1. How to solve and realize the general spectral Nevanlinna-Pick interpolation problem?

2.Does the existence of φ depend sufficiently on the equality of Carathéodory and Kobayashi distances?

3.How to use the above results and problems to design the spectral controller for uncertain system?

Appendix **Geometric Layout of Γ**

The following two figures are the special case of Γ. We can see that they are non-convex, non-smooth but bounded.

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References

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